Density Functions

Distinguishable Non-interacting Point Particles

In the simplest case, consider N distinguishable non-interacting point particles and suppose the probability of finding them in the configuration $\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_N\}$ is given by $P(\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_N)$, where $\vec{x_i}$ is the complete set of coordinates that describe the state of particle i that may include spatial, spin or momentum information or what not. The probability density of finding particle 1 in the region $\vec{x}$ is then

$$p_1^{(1)}(\vec{x}) = \int P(\vec{x}, \vec{x}_2, \cdots, \vec{x}_N) d\vec{x}_2 d\vec{x}_3 \cdots d\vec{x}_N \tag{1}$$

Following the above example we can then define ***single*** *particle probability density* functions $p_1^{(i)}(\vec{x})$ for each particle i. In general $p_1^{(i)} \neq p_1^{(j)}$ for $i \neq j$, but we do have[1]

$$\int p_1^{(i)}(\vec{x}) d\vec{x} = 1 \ \forall i \tag{2}$$

For a pair of particles

$$p_2^{(1,2)}(\vec{x}_1, \vec{x}_2) = \int P(\vec{x}_1, \vec{x}_2, \vec{x}_3, \cdots, \vec{x}_N) d\vec{x}_3 \cdots d\vec{x}_N \tag{3}$$

Notice $p_2^{(i,j)}$ is only defined for $i \neq j$. We then have the comforting reduction

$$\boxed{\int p_2^{(i,j)}(\vec{x}_1, \vec{x}_2) d\vec{x}_2 = p_1^{(i)}(\vec{x}_1)} \tag{4}$$

Further, since the particles are non-interacting, we have

$$P(\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_N) = p_1^{(1)}(\vec{x}_1) p_1^{(2)}(\vec{x}_2) \cdots p_1^{(N)}(\vec{x}_N) \tag{5}$$

and $p_2^{(i,j)}$ reduces to

$$\boxed{p_2^{(i,j)}(\vec{x}_1, \vec{x}_2) = p_1^i(\vec{x}_1) p_1^j(\vec{x}_2)} \tag{6}$$

The nice reduction formula (4) and (6) define our intuitive understanding of particles. Indeed they describe the probabilistic properties of a table of ghost billiard balls (they just go through each other) and are easy to visualize. (4) says the probability of having ball i at $\vec{x}_1$ is equal to the sum of probabilities of having ball i at $\vec{x}_1$ and ball j at $\vec{x}_2$ over all possible $\vec{x}_2$. (6) says the probability of having ball i at $\vec{x}_1$ and ball j at $\vec{x}_2$ is equal to the product of probabilities of having ball i at $\vec{x}_1$ with ball j being anywhere and the probability having ball j at $\vec{x}_2$ with ball i being anywhere. Notice (4) doesn't require non-interaction, whereas (6) does.

For a smooth transition into indistinguishable particles, let's study distinguishable particles while pretending they're indistinguishable. This is a subtle but very important exercise. It's tempting to think if we take the previously distinguishable ghost billiard balls, paint them all black and carry out

[1] Only if $\vec{x_i}$ is a complete set of coordinates for each i

the aforementioned calculations then we can get the properties of indistinguishable particles. This line of reasoning is wrong. Distinguishable particles and indistinguishable particles have fundamentally different statistics.

Let's still consider N distinguishable particles, but now ask what the probability density of finding a ball with ANY label around $\vec{x}$. This *particle density*

$$\rho_1(\vec{x}) = \sum_{i=1}^{N} p_1^{(i)}(\vec{x}) \tag{7}$$

and *pair density*

$$\rho_2(\vec{x}_1, \vec{x}_2) = \sum_{i \neq j} p_2^{(i,j)}(\vec{x}_1, \vec{x}_2) \tag{8}$$

Does $\rho_2(\vec{x}_1, \vec{x}_2)$ still reduce to $\rho_1(\vec{x}_1)$ when $\vec{x}_2$ is integrated out?

$$\boxed{\int \rho_2(\vec{x}_1, \vec{x}_2) d\vec{x}_2} = \sum_{i \neq j} \int p_2^{(i,j)}(\vec{x}_1, \vec{x}_2) d\vec{x}_2$$
$$= \sum_{i \neq j} p_1^{(i)}(\vec{x}_1) = (N-1) \sum_{i} p_1^{(i)}(\vec{x}_1) \boxed{= (N-1)\rho_1(\vec{x}_1)} \tag{9}$$

Unfortunately, no, but can it be split into a product of particle densities?

$$\begin{aligned} \rho_2(\vec{x}_1, \vec{x}_2) &= \sum_{i \neq j} p_1^{(i)}(\vec{x}_1) p_1^{(j)}(\vec{x}_2) \\ &= \sum_{i} p_1^{(i)}(\vec{x}_1) \sum_{j \neq i} p_1^{(j)}(\vec{x}_2) \\ &= \sum_{i} p_1^{(i)}(\vec{x}_1) \left(\sum_{j} p_1^{(j)}(\vec{x}_2) - p_1^{(i)}(\vec{x}_2) \right) \\ &= \rho_1(\vec{x}_1)\rho_1(\vec{x}_2) - \sum_{i} p_1^{(i)}(\vec{x}_1) p_1^{(i)}(\vec{x}_2) \end{aligned} \tag{10}$$

Notice if $\vec{x}_1 \neq \vec{x}_2$, $p_1^{(i)}(\vec{x}_1)p_1^{(i)}(\vec{x}_2) = 0$ since particle i can't be at two places at once, also $\int p_1^{(i)}(\vec{x}_1)p_1^{(i)}(\vec{x}_2)d\vec{x}_2 = 1$, thus $p_1^{(i)}(\vec{x}_1)p_1^{(i)}(\vec{x}_2) = \delta(\vec{x}_1 - \vec{x}_2)$ and

$$\boxed{\rho_2(\vec{x}_1, \vec{x}_2) = \rho_1(\vec{x}_1)(\rho_1(\vec{x}_2) - \delta(\vec{x}_1 - \vec{x}_2))} \tag{11}$$

The non-intuitive reduction formula (9) and (11) are what makes dealing with densities without label confusing. However, we can actually interpret (11) quite nicely by introducing the *hole density*

$$h(\vec{x}_1, \vec{x}_2) \equiv \frac{\rho_2(\vec{x}_1, \vec{x}_2)}{\rho_1(\vec{x}_1)} - \rho_1(\vec{x}_2) = -\delta(\vec{x}_1 - \vec{x}_2) \tag{12}$$

This way

$$\rho_2(\vec{x}_1, \vec{x}_2) = \rho_1(\vec{x}_1)(\rho_1(\vec{x}_2) + h(\vec{x}_1, \vec{x}_2)) \tag{13}$$

That is, when a particle is set at $\vec{x}_1$. To calculate its pair density with another particle at $\vec{x}_2$, we need to modulate the probability density by a hole density $h(\vec{x}_1, \vec{x}_2)$. Of course, in this case we know exactly where this hole came from. When we remove a particle from our pool of N particles, the density of particles it can form a pair with $\rho_1(\vec{x}_2) + h(\vec{x}_1, \vec{x}_2)$ must normalize to $N-1$.