Wronskian

Linear Dependence

If a set of n solutions $\{y_i(x)\}$ to the differential equation

$$p_0 y^{(n)}(x) + p_1 y^{(n-1)}(x) + \cdots + p_n y(x) = 0 \tag{1}$$

are linearly dependent, $\exists\{\lambda_i\}$ not all zero, such that

$$\lambda_1 y_1(x) + \lambda_2 y_2(x) + \cdots + \lambda_n y_n(x) = 0 \tag{2}$$

since $\frac{d}{dx}$ is linear

$$\begin{pmatrix} y_1(x) & y_2(x) & \cdots & y_{n-1}(x) & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_{n-1}'(x) & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ y_1^{(n-2)}(x) & y_2^{(n-2)}(x) & \cdots & y_{n-1}^{(n-2)}(x) & y_n^{(n-2)}(x) \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_{n-1}^{(n-1)}(x) & y_n^{(n-1)}(x) \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-1} \\ \lambda_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \tag{3}$$

this implies

$$\begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_{n-1}(x) & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_{n-1}'(x) & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ y_1^{(n-2)}(x) & y_2^{(n-2)}(x) & \cdots & y_{n-1}^{(n-2)}(x) & y_n^{(n-2)}(x) \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_{n-1}^{(n-1)}(x) & y_n^{(n-1)}(x) \end{vmatrix} = 0 \tag{4}$$

Definition

For a set of functions $y_1(x), y_2(x), \cdots, y_n(x)$, the *Wronskian* is defined as

$$W[\{y_i\}] \equiv \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_{n-1}(x) & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_{n-1}'(x) & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ y_1^{(n-2)}(x) & y_2^{(n-2)}(x) & \cdots & y_{n-1}^{(n-2)}(x) & y_n^{(n-2)}(x) \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_{n-1}^{(n-1)}(x) & y_n^{(n-1)}(x) \end{vmatrix} \tag{5}$$

Property

Determinant has the nice property that it's linear in any single row and that it returns zero when any two rows are linearly dependent. Thus

$$\frac{dW}{dx} = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_{n-1}(x) & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_{n-1}'(x) & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ y_1^{(n-2)}(x) & y_2^{(n-2)}(x) & \cdots & y_{n-1}^{(n-2)}(x) & y_n^{(n-2)}(x) \\ y_1^{(n)}(x) & y_2^{(n)}(x) & \cdots & y_{n-1}^{(n)}(x) & y_n^{(n)}(x) \end{vmatrix} \tag{6}$$

If $\{y_i(x)\}$ are all solutions to (1), then

$$y_i^{(n)}(x) = -\frac{1}{p_0}\left(p_1 y_i^{(n-1)}(x) + \cdots + p_n y_i(x)\right) \tag{7}$$

and using the linearity of determinant

$$\frac{dW}{dx} = -\frac{p_1}{p_0}W \tag{8}$$

Thus

$$W[\{y_i\}; x] = W[\{y_i\}; x_0]e^{-\int_{x_0}^{x}\frac{p_1(\xi)}{p_0(\xi)}d\xi} \tag{9}$$

Therefore, W is either never zero, or uniformly zero. Further

$$\boxed{\text{ODE and } W = 0 \Rightarrow \{y_i\} \text{ are linearly dependent}} \tag{10}$$