Wave Equation

The wave equation

$$\frac{\partial^2\phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0 \tag{1}$$

with the initial values (Cauchy data) $\phi(x,0), \dot{\phi}(x,0)$ is most easily solved through factorization

$$(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t})(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t})\phi = 0 \tag{2}$$

However, historically the first people to solve (1) had slightly different interpretations of (2). Euler interpreted it geometrically while d'Alembert interpreted it algebraically.

Euler's solution

Euler wants to see what the left most operator in (2) does geometrically. He realized that if we can find a parametrization $x(s_1), t(s_1)$ such that

$$\left\{\begin{array}{l} \frac{\partial x}{\partial s_1} = 1 \\ \frac{\partial t}{\partial s_1} = \frac{1}{c} \end{array}\right. \Rightarrow \left\{\begin{array}{l} x = s_1 + a \\ t = \frac{1}{c}s_1 + b \end{array}\right. \tag{3}$$

then we may rewriting (2) as

$$\boxed{\frac{\partial}{\partial s_1}A(s_1) = 0} \tag{4}$$

Voila, the operator $\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}$ does not change the value of its operand along the curve $\gamma(s_1) = (x(s_1), t(s_1))$. Therefore we can calculate the value of its operand at any point (x, y) by tracing back to the Cauchy data, indeed, using $s_1' = -cb$ we have

$$A(x(s_1'), t(s_1')) = A(a - cb, 0) = A(x(s_1) - ct(s_1), 0) \Rightarrow$$
$$\boxed{A(x,t) = A(x - ct, 0)} \tag{5}$$

where I have obtained from (3) that

$$\left\{\begin{array}{l} a = x - s_1 \\ b = t - \frac{1}{c}s_1 \end{array}\right. \tag{6}$$

Reverse the order of factorization of (2), we can define s_2 such that

$$\frac{\partial}{\partial s_2} B(s_2) = 0 \tag{7}$$

with

$$\boxed{B(x,t) = B(x+ct,0)} \tag{8}$$

Writing out A, B explicitly

$$\begin{cases} A = \phi' - \frac{1}{c}\dot{\phi} \\ B = \phi' + \frac{1}{c}\dot{\phi} \end{cases} \tag{9}$$

we have

$$\begin{aligned} \dot{\phi}(x,t) =& \frac{c}{2}\left\{B(x,t) - A(x,t)\right\} \\ =& \frac{c}{2}\left\{B(x+ct,0) - A(x-ct,0)\right\} \end{aligned} \tag{10}$$

which is a known function. The solution can therefore obtained by integrating over time

$$\phi(x,T) = \phi(x,0) + \int_0^T \dot{\phi}(x,t)dt \tag{11}$$

This is a non-trivial integral, but can be done with lots of change of variable (which results in change of integration limits) and the final form is

$$\boxed{\phi(x,T) = \frac{1}{2}\left\{\phi(x+ct,0) + \phi(x-ct,0)\right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \dot{\phi}(\xi,0)d\xi} \tag{12}$$

Euler's solution gives nice geometrical interpretations to the operators and shows that the final solution only depends on Cauchy data that can be traced back within a light cone. However, it is mathematically tedious.

d'Alembert's solution

d'Alembert starts by changing to the light-cone coordinates

$$\left\{ \begin{array}{l} \xi = x + ct \\ \eta = x - ct \end{array} \right. \Rightarrow \left\{ \begin{array}{l} x = \frac{1}{2}(\xi + \eta) \\ ct = \frac{1}{2}(\xi - \eta) \end{array} \right. \tag{13}$$

this way we may rewrite (2) as

$$4\frac{\partial^2}{\partial\xi\partial\eta}\phi(\xi,\eta) = 0 \tag{14}$$

the general solution is therefore

$$\phi(\xi,\eta) = f(\xi) + g(\eta) \tag{15}$$

or

$$\phi(x,t) = f(x+ct) + g(x-ct) \tag{16}$$

now the initial values say

$$\left\{ \begin{array}{l} \phi(x,0) = f(x) + g(x) \\ \dot{\phi}(x,0) = c(f'(x) - g'(x)) \end{array} \right. \Rightarrow$$
$$\left\{ \begin{array}{l} f(x) + g(x) = \phi(x,0) \\ f(x) - g(x) = \frac{1}{c}\int_0^x \dot{\phi}(\xi,0)d\xi + C \end{array} \right. \Rightarrow$$
$$\left\{ \begin{array}{l} f(x) = \frac{1}{2}\phi(x,0) + \frac{1}{2c}\int_0^x \dot{\phi}(\xi,0)d\xi + \frac{1}{2}C \\ g(x) = \frac{1}{2}\phi(x,0) - \frac{1}{2c}\int_0^x \dot{\phi}(\xi,0)d\xi - \frac{1}{2}C \end{array} \right. \tag{17}$$

and we have obtained our solution

$$\phi(x,t) = \frac{1}{2}\left\{\phi(x+ct,0) + \phi(x-ct)\right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \dot{\phi}(\xi,0)d\xi \tag{18}$$

Notice, this is the same as Euler's solution, but the steps are a lot more clearly laid out. It seems a bit like cheating though since the light-cone coordinates came out of nowhere. (I'm pretty sure he got the inspiration from Euler)