Reduction of Order

Reduction of order is really a misnomer. It refers to the process of eliminating the $n-1^{\text{th}}$ order term in an n^{th} order ODE by a clever transformation. By definition, this process *does not* change the order of the ODE, it merely makes it look cleaner.

Normal Form

Given an n^{th} order differential equation of the form

$$p_0 y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = 0 \tag{1}$$

We can remove its $(n-1)^{\text{th}}$ order term by choosing

$$\begin{cases} y = \omega\tilde{y} \\ \omega' = -\frac{p_1}{np_0}\omega \end{cases} \tag{2}$$

To see this, plug $y = \omega\tilde{y}$ into (1)

$$p_0\omega\tilde{y}^{(n)} + (np_0\omega' + p_1\omega)\tilde{y}^{(n-1)} + \cdots + p_n\omega\tilde{y} = 0 \tag{3}$$

the $n-1^{\text{th}}$ order term $np_0\omega' + p_1\omega$ is, because of the choice of ω, naturally 0. This is especially useful for a second-order equation since it reduces to a 1D Schrödinger-like equation and may be solved via physical intuition. **Example** Given a general second-order equation

$$p_0 y'' + p_1 y' + p_2 y = 0 \tag{4}$$

by choosing, we can write an equation in $\tilde{y}$ that's equivalent to (4)

$$\boxed{\begin{cases} y = \omega\tilde{y} \\ \omega' = -\frac{p_1}{2p_0}\omega \end{cases}} \tag{5}$$

$$\begin{aligned} p_0\omega\tilde{y}'' + (2p_0\omega' + p_1\omega)\tilde{y}' + (p_0\omega'' + p_1\omega' + p_2\omega)\tilde{y} = 0 \Rightarrow \\ p_0\omega\tilde{y}'' + \left(-p_0(\frac{p_1}{2p_0})'\omega - \frac{1}{2}p_1\omega' + p_2\omega\right)\tilde{y} = 0 \Rightarrow \\ p_0\tilde{y}'' + \left(p_2 - p_0(\frac{p_1}{2p_0})' - \frac{p_1^2}{4p_0}\right)\tilde{y} = 0 \end{aligned} \tag{6}$$

If we further have $p_0' = 0$, then

$$\boxed{p_0\tilde{y}'' + \left(p_2 - \frac{1}{2}p_1' - \frac{1}{4}\frac{p_1^2}{p_0}\right)\tilde{y} = 0} \tag{7}$$

Sturm-Liouville Form

If we want to write (4) in Sturm-Liouville form

$$\boxed{\frac{1}{w}(wq_0y')' + q_2y = \frac{1}{w}\frac{d}{dx}wq_0\frac{d}{dx}y + q_2y = 0} \tag{8}$$

we must choose $q_0 = p_0$, $q_2 = p_2$ and the weight function w such that

$$\boxed{\frac{w'}{w}p_0 + p_0' = p_1 \Rightarrow w = \frac{1}{p_0}e^{\int_a^x (\frac{p_1(\xi)}{p_0(\xi)})d\xi}} \tag{9}$$

To see this, expand (8)

$$\begin{aligned} q_0y'' + \frac{1}{w}(w'q_0 + wq_0')y' + q_2y = 0 \Rightarrow \\ q_0y'' + (\frac{w'}{w}q_0 + q_0')y' + q_2y = 0 \Rightarrow \\ p_0y'' + p_1y' + p_2y = 0 \end{aligned} \tag{10}$$

This is an important result, because the Sturm-Liouville operator

$$\hat{L} = \frac{1}{w}\frac{d}{dx}wp_0\frac{d}{dx} + p_2 \tag{11}$$

happen to be formally self-adjoint given the weight $w = \frac{1}{p_0}exp\left(\int_a^x \frac{p_1}{p_0}\right)$. That is, we can eye-ball the weight that will make a Sturm-Liouville operator formally self-adjoint by putting it into the form (11).