Potential Theory

Potential theory involves Laplacian (in this case Poisson) problems with Dirichlet or Neumann boundary conditions

$$\begin{cases} -\nabla^2\phi(\vec{r}) = f(\vec{r}) & \vec{r} \in \Omega \\ \phi(\vec{r}) = g_o(\vec{r}) & \vec{r} \in \partial\Omega \end{cases} \quad \begin{cases} -\nabla^2\phi(\vec{r}) = f(\vec{r}) & \vec{r} \in \Omega \\ \hat{n}\cdot\vec{\nabla}\phi(\vec{r}) = g(\vec{r}) & \vec{r} \in \partial\Omega \end{cases} \tag{1}$$

A vector identity and divergence theorem will help us crack potential theory

$$\vec{\nabla}\cdot(\chi\vec{\nabla}\phi) = \vec{\nabla}\chi\cdot\vec{\nabla}\phi + \chi\nabla^2\phi \tag{2}$$

$$\int_\Omega d^n r \left\{\vec{\nabla}\cdot\vec{A}\right\} = \int_{\partial\Omega} d^{n-1}r\left\{\hat{n}\cdot\vec{A}\right\} \tag{3}$$

where $\chi(\vec{r}), \phi(\vec{r})$ are potentials (scalar fields) and $\vec{A}(\vec{r})$ is a vector field.

Dirichlet's Principle

Dirichlet suggested that the potential problem (1) can be converted into a variation problem by defining the functional

$$J[\chi] = \int_\Omega d^n r \left\{\frac{1}{2}|\vec{\nabla}\chi|^2 - \chi f\right\} \tag{4}$$

He claimed that the function that minimizes J is ϕ, the solution to (1). To verify this, let's first simplify J with our best friends (2) and (3)

$$\begin{aligned} J[\chi] &= \int_\Omega d^n r\left\{\frac{1}{2}\vec{\nabla}\chi\cdot\vec{\nabla}\chi + \chi\nabla^2\phi\right\} \\ &= \int_\Omega d^n r\left\{\frac{1}{2}\vec{\nabla}\chi\cdot\vec{\nabla}\chi + \vec{\nabla}\cdot(\chi\vec{\nabla}\phi) - \vec{\nabla}\chi\cdot\vec{\nabla}\phi\right\} \\ &= \int_{\partial\Omega} d^{n-1}r\left\{\chi\hat{n}\cdot\vec{\nabla}\phi\right\} + \int_\Omega d^n r\left\{\frac{1}{2}\vec{\nabla}\chi\cdot\vec{\nabla}\chi - \vec{\nabla}\chi\cdot\vec{\nabla}\phi\right\} \end{aligned} \tag{5}$$

with the solution ϕ plugged in, the value of the functional takes the form

$$J[\phi] = \int_{\partial\Omega} d^{n-1}r\left\{\phi\hat{n}\cdot\vec{\nabla}\phi\right\} - \int_\Omega d^n r\left\{\frac{1}{2}|\vec{\nabla}\phi|^2\right\} \tag{6}$$

The difference is then

$$\begin{aligned} J[\chi] - J[\phi] &= \int_{\partial\Omega} d^{n-1}r\left\{(\chi-\phi)\hat{n}\cdot\vec{\nabla}\phi\right\} + \frac{1}{2}\int_\Omega d^n r\left\{|\vec{\nabla}\chi|^2 - 2\vec{\nabla}\chi\cdot\vec{\nabla}\phi + |\vec{\nabla}\phi|^2\right\} \\ &= \int_{\partial\Omega} d^{n-1}r\left\{(\chi-\phi)\hat{n}\cdot\vec{\nabla}\phi\right\} + \frac{1}{2}\int_\Omega d^n r\left\{|\vec{\nabla}(\chi-\phi)|^2\right\} \end{aligned} \tag{7}$$

and we finally see that the potential problem is equivalent to the variational problem as long as the boundary term vanishes. With Dirichlet boundary, this is only true if the variation is made within the set of functions with the correct boundary condition $\chi(\vec{r}) = \phi(\vec{r}) = g(\vec{r}), \forall \vec{r} \in \partial\Omega$. With Neumann boundary, we don't need to restrict the range of functions, but we do need the boundary condition to be uniformly 0, because that way $\hat{n} \cdot \vec{\nabla}\phi(\vec{r}) = g(\vec{r}) = 0, \forall \vec{r} \in \partial\Omega$ and the boundary term vanishes.

Dirmann's Principle[1]

The *Dirichlet's energy* defined by (4) does not satisfactorily solve the potential problem with Neumann boundary (it requires $g = 0$). Fortunately, we can define a different functional that compliments the solution given above.

$$K[\chi] = \int_\Omega d^n r \left\{ \frac{1}{2} |\vec{\nabla}\chi|^2 \right\} - \int_{\partial\Omega} \chi g d^{n-1} r \tag{8}$$

Let's play the same game

$$\begin{aligned} K[\chi] &= \int_\Omega d^n r \left\{ \frac{1}{2} |\vec{\nabla}\chi|^2 \right\} - \int_{\partial\Omega} \chi(\hat{n} \cdot \vec{\nabla}\phi) d^{n-1} r \\ &= \int_\Omega d^n r \left\{ \frac{1}{2} |\vec{\nabla}\chi|^2 - \vec{\nabla} \cdot (\chi \vec{\nabla}\phi) \right\} \\ &= \int_\Omega d^n r \left\{ \frac{1}{2} |\vec{\nabla}\chi|^2 - \vec{\nabla}\chi \cdot \vec{\nabla}\phi - \chi \nabla^2 \phi \right\} \qquad (9) \\ & \qquad (10) \end{aligned}$$

$$K[\phi] = - \int_\Omega d^n r \left\{ \frac{1}{2} |\vec{\nabla}\phi|^2 - \phi \nabla^2 \phi \right\} \tag{11}$$

$$K[\chi] - K[\phi] = \frac{1}{2} \int_\Omega d^n r \left\{ |\vec{\nabla}(\chi - \phi)|^2 - (\chi + \phi) \nabla^2 \phi \right\} \tag{12}$$

If $\nabla^2 \phi = f = 0$, then ϕ would minimize K. Voila! We have converted the potential problem to a variation problem by JK[2].

[1] I made the name up
[2] Pun totally intended