Ocean Waves

Original Non-Linear Problem

The velocity field in an irrotational fluid ($\vec{\nabla} \times \vec{v} = \vec{0}$) can be expressed as the gradient of a scalar field $\vec{v} = \vec{\nabla}\phi$. If the fluid is in addition incompressible, then Laplace's equation must be satisfied in the bulk

$$\nabla^2 \phi = 0 \tag{1}$$

Recall from Chapter 1, the boundary conditions that simulate ocean waves

$$\begin{cases} \frac{\partial \phi}{\partial y} = 0 & y = 0 \\ \frac{\partial \phi}{\partial t} + \frac{1}{2}|\vec{\nabla}\phi|^2 + gh = 0 & y = h \\ \frac{\partial h}{\partial t} - \frac{\partial \phi}{\partial y} + \frac{\partial h}{\partial x}\frac{\partial \phi}{\partial x} = 0 & y = h \end{cases} \tag{2}$$

where the first condition states that water cannot flow through the ocean floor, the second says pressure is constant on the ocean surface and the last says surface fluid remains on the surface. The first condition is easily satisfied by writing down a particular solution to Laplace's equation in the form

$$\phi = \cosh(ky)\cos(kx - \omega t) \tag{3}$$

Linearization and Deep Water

The last two boundary conditions are harder to satisfy, but for waves with small amplitudes we may linearize them by getting rid of any term with a product of derivatives

$$\begin{cases} \frac{\partial \phi}{\partial t} + gh = 0 & y = h \\ \frac{\partial h}{\partial t} - \frac{\partial \phi}{\partial y} = 0 & y = h \end{cases} \Rightarrow$$

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial y} = 0,\ y = h \tag{4}$$

If we further suppose the wave amplitudes are small enough that the boundary conditions roughly apply to the equilibrium height $y = h_o$, we may plug (3) in (4) to get

$$-\omega^2 \cosh(kh_o)\cos(kx - \omega t) + kg\sinh(kh_o)\cos(kx - \omega t) = 0 \Rightarrow$$

$$\omega^2 = gk\tanh(kh_o) \tag{5}$$

In the deep water limit ($kh_o \gg 1$)

$$\lim_{h_o \to \infty} \omega = \sqrt{gk} \tag{6}$$

and since $y \sim h_o$

$$\cosh(ky) = e^{k(y-h_o)} \frac{1 + e^{-2ky}}{2e^{-kh_o}} \approx e^{k(y-h_o)} \tag{7}$$

thus the solution

$$\boxed{\phi(x, y) = e^{k(y-h_o)} \cos(kx - \omega t)} \tag{8}$$

where $\omega = \sqrt{gk}$. If we further assume the motion of water is localized near the surface we can even get the wave profile by solving

$$\begin{gathered}
\vec{v} = \vec{\nabla}\phi|_{x=x_o, y=h_o} \Rightarrow \\
\begin{cases} \frac{dx}{dt} = -k\sin(kx_o - \omega t) \\ \frac{dy}{dt} = k\cos(kx_o - \omega t) \end{cases} \Rightarrow \\
\begin{cases} x = x_o - \frac{k}{\omega}\cos(kx_o - \omega t) \\ y = y_o - \frac{k}{\omega}\sin(kx_o - \omega t) \end{cases} \Rightarrow \\
\boxed{\begin{cases} x = x_o - \sqrt{\frac{k}{g}}\cos(kx_o - \sqrt{gk}t) \\ y = y_o - \sqrt{\frac{k}{g}}\sin(kx_o - \sqrt{gk}t) \end{cases}}
\end{gathered} \tag{9}$$

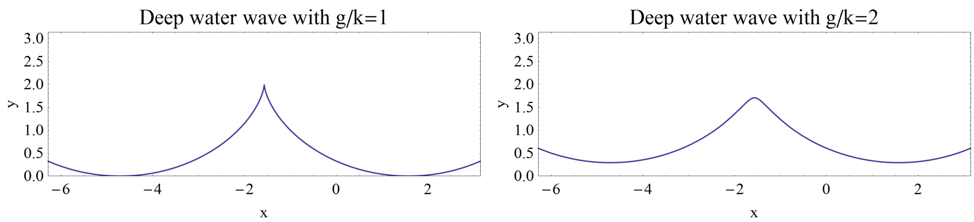

Figure 1: Deep Water wave has a cusp when $g = k$