Noether's Theorem

We've seen how difficult it could be to find a conservation law from the equations of motion. Noether realized the conservation laws are hidden in the action integral and symmetry gives us a nice way to find them.

Rotation Symmetry and Angular Momentum

Consider the central force action integral

$$S[r,\theta] = \int_{t_1}^{t_2} \mathscr{L}(r,\dot{r},\dot{\theta})dt = \int_{t_1}^{t_2} \{\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)\}dt \tag{1}$$

Notice the rotation symmetry - when $\theta(t) \to \theta(t) + \epsilon$, S is unchanged. In other words, $\delta S = 0$ when $\delta\theta = \epsilon$ is independent of t. If $\theta(t)$ obeys the equations of motion, δS will be stationary against ANY fixed-end variation. Therefore we can extend our luck and consider a time-dependent fixed-end variation $\delta\theta = \epsilon(t)$. This way *something else* better make $\delta S = 0$ (spoiler alert: that something else is the conservation law). By definition

$$\begin{aligned}\delta S &= \int_{t_1}^{t_2} \frac{\delta\mathscr{L}}{\delta\theta}\delta\theta dt = \int_{t_1}^{t_2} (\frac{\partial\mathscr{L}}{\partial\theta} - \frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{\theta}})\delta\theta dt \\ &= -\int_{t_1}^{t_2} \frac{d}{dt}(mr^2\dot{\theta})\epsilon dt \end{aligned}\tag{2}$$

If $\theta(t)$ satisfies the equations of motion $\delta S = 0$ thus

$$\boxed{\frac{d}{dt}(mr^2\dot{\theta}) = 0} \tag{3}$$

This example is kindda dump, because it ends up being a restatement of the θ equation of motion, but it does demonstrate the search process suggested by Noether. Let's look at a less trivial example.

Time Translation and Energy

Suppose an action integral

$$S[q] = \int_{t_1}^{t_2} \mathscr{L}(q,\dot{q})dt \tag{4}$$

is time translation invariant. That is, when $q(t) \to q(t+\epsilon) = q + \epsilon\dot{q}$, $\delta S = 0$. In other words, $\delta S = 0$ when $\delta q = \epsilon\dot{q}$. Again, allow fixed-end $\delta q = \epsilon(t)\dot{q}$

$$\delta S = \int_{t_1}^{t_2} (\frac{\partial \mathscr{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}})(\epsilon\dot{q})dt \tag{5}$$

when $q(t)$ satisfies the equations of motion, $\delta S = 0$ and we'd better have

$$\begin{aligned} \dot{q}\frac{\partial \mathscr{L}}{\partial q} - \dot{q}\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}} &= 0 \Rightarrow \\ \dot{q}\frac{\partial \mathscr{L}}{\partial q} + \ddot{q}\frac{\partial \mathscr{L}}{\partial \dot{q}} - \frac{d}{dt}(\dot{q}\frac{\partial \mathscr{L}}{\partial \dot{q}}) &= 0 \Rightarrow \\ \boxed{\frac{d}{dt}(\mathscr{L} - \dot{q}\frac{\partial \mathscr{L}}{\partial \dot{q}}) = 0} \end{aligned} \tag{6}$$

Lo and behold, action principle plus time translation symmetry implies Hamiltonian/Energy is constant in time.