Landau Levels

Particle in Magnetic Field

With an external magnetic field the free Hamiltonian $H_o = \frac{p^2}{2m}$ becomes

$$H = \frac{\pi^2}{2m} = \frac{(\vec{p} - q\vec{A})^2}{2m} \tag{1}$$

where $\vec{A}$ is the vector potential that defines the magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$. Choosing the Landau gauge $\vec{A} = B_o x \hat{y}$ for $\vec{B} = B_o \hat{z}$, we have

$$H = \frac{p^2}{2m} - \frac{qB_o p_y}{m} x + \frac{q^2 B_o^2}{2m} x^2 \tag{2}$$

If the particles are constraint to move in the $x - y$ plane, the ansatz

$$\psi_{p_y} = e^{\frac{i p_y y}{\hbar}} \phi_{p_y}(x), \ \ p_y = \hbar k_y \tag{3}$$

is valid since H is translationally invariant in the y direction. Thus by

$$\begin{aligned} H\psi_{p_y} &= E(p_y)\psi_{p_y} \Rightarrow \\ \left(\frac{p_x^2}{2m} + \frac{p_y^2}{2m} - \frac{qB_o p_y}{m} x + \frac{q^2 B_o^2}{2m} x^2\right)\phi_{p_y} &= E(p_y)\phi_{p_y} \Rightarrow \\ \left(\frac{p_x^2}{2m} + \frac{(qB_o)^2}{2m}\left(\frac{\hbar^2 k_y^2}{(qB_o)^2} - 2\frac{\hbar k_y}{qB_o} x + x^2\right)\right)\phi_{p_y} &= E(p_y)\phi_{p_y} \end{aligned} \tag{4}$$

Define $\ell_B^2 \equiv \frac{\hbar}{qB}$, $\omega_c \equiv \frac{|qB_o|}{m}$ and complete the square

$$\frac{1}{2m}\left(p_x^2 + m^2 \omega_c^2 (x - k_y \ell_B^2)^2\right)\phi_{p_y} = E(p_y)\phi_{p_y} \tag{5}$$

This is a harmonic oscillator at $x = k_y \ell_B^2$ with energy levels

$$\boxed{E_n = \hbar\omega_c(n + \frac{1}{2})} \tag{6}$$

And the final wave function

$$\boxed{\psi_{n,p_y} = e^{i k_y y} H_n(x - k_y \ell_B^2) e^{-\frac{(x - k_y \ell_B^2)^2}{4\ell_B^2}}} \tag{7}$$

where H_n are the Hermite polynomials. The energy levels (6) are called *Landau levels*. There are many quantum states for every Landau level i.e. for a given n, every p_y corresponds to a state with the same energy E_n.

Number of States

Suppose the system is of size $L_x \times L_y$, then the separation between harmonic oscillators

$$\Delta x = \Delta k_y \ell_B^2 = (\frac{2\pi}{L_y})\ell_B^2 \tag{8}$$

Thus the number of oscillators we can fit into the system

$$\boxed{N = \frac{L_x}{\Delta x} = \frac{L_x L_y}{2\pi \ell_B^2}} \tag{9}$$

Had we chosen a different gauge, say $\vec{A} = -B_o y\hat{x}$, then we would have had

$$N = \frac{L_y}{\Delta y} = \frac{L_x L_y}{2\pi \ell_B^2} \tag{10}$$

Plugging in $\ell_B^2 \equiv \frac{\hbar}{qB}$ we see that for electrons

$$N = \frac{q}{\hbar} B L_x L_y = \frac{B L_x L_y}{\hbar/e} = \frac{\phi}{\phi_o} \tag{11}$$

N is the number of flux quanta, which is a measurable quantity. It is comforting that N is gauge invariance, since physical quantities should obey the same physics regardless of description. The choice of gauge is a choice of description of the physics (much like coordinates), but this choice of description should not affect the result of the underlying physics.