Integral Transform

Various integral transforms such as Fourier transform and Laplace transform are our best friends when trying to solve integral equations. Fourier transform solves Type I inhomogeneous Fredholm equations whereas Laplace transform solves Type I inhomogeneous Volterra problems, but the integral kernel must be translation invariant ($K(x,y) = K(x-y)$) aka the integral equation takes the form of a convolution

Fourier Transform

Consider a Type II inhomogeneous Fredholm equation

$$u(x) = f(x) + \int_{-\infty}^{\infty} \bar{K}(x,y)u(y)dy \tag{1}$$

we can rewrite it as Type I using a delta function

$$\int_{-\infty}^{\infty} \left\{\delta(x-y) - \bar{K}(x,y)\right\} u(y)dy = f(x) \tag{2}$$

If the *integral kernel*

$$K(x,y) = \delta(x-y) - \bar{K}(x,y) \tag{3}$$

satisfies $K(x,y) = K(x-y)$, we may Fourier transform both sides of (2)

$$\begin{aligned}\int_{-\infty}^{\infty} dxe^{-ikx} \int_{-\infty}^{\infty} dy \left\{K(x-y)u(y)\right\} = \int_{-\infty}^{\infty} dxe^{-ikx} \left\{f(x)\right\} \Rightarrow \\ \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left\{e^{-ik(x-y)}K(x-y)e^{-iky}u(y)\right\} = \tilde{f}(k) \Rightarrow \\ \tilde{K}(k)\tilde{u}(k) = \tilde{f}(k)\end{aligned} \tag{4}$$

and the solution can be obtained from an inverse Fourier transform

$$\boxed{u(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \frac{\tilde{f}(k)}{\tilde{K}(k)}} \tag{5}$$

Laplace Transform

Consider a Type II inhomogeneous Volterra equation rewritten as

$$\int_{0}^{x} K(x,y)u(y)dy = f(x) \tag{6}$$

If $K(x,y)=K(x-y)$, we can Laplace transform both sides

$$\int_0^\infty dx e^{-px}\int_0^x dy\,\{K(x-y)u(y)\}=\tilde{f}(p) \tag{7}$$

make the change of variable $\xi=x-y,\eta=y$ then (7) becomes

$$\begin{aligned}\int_0^\infty d\xi\int_0^\infty d\eta\left\{e^{-p\xi}K(\xi)e^{-p\eta}u(\eta)\right\}=\tilde{f}(p)\Rightarrow\\ \tilde{K}(p)\tilde{u}(p)=\tilde{f}(p)\end{aligned} \tag{8}$$

and the solution can be obtained through an inverse transform

$$\boxed{u(x)=\mathcal{L}^{-1}\left(\frac{\tilde{f}(p)}{\tilde{K}(p)}\right)=\frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT}dpe^{px}\frac{\tilde{f}(p)}{\tilde{K}(p)}} \tag{9}$$

This is an ugly contour integral that tends to be difficult to evaluate. Fortunately, the given Volterra equation (6) satisfies $f(0)=0$, thus laplace transform can change a derivative to multiplication by p

$$\begin{aligned}\widetilde{f'}(p)=&\int_0^x e^{-px}\frac{df}{dx}dx=e^{-px}f(x)|_0^\infty+p\int_0^x e^{-px}f dx\\ =&p\int_0^x e^{-px}f dx=p\tilde{f}(p)\end{aligned} \tag{10}$$

Thus (9) becomes

$$u(x)=\mathcal{L}^{-1}\left(\frac{1}{p\tilde{K}(p)}\widetilde{f'}(p)\right) \tag{11}$$

with any luck, the fomula for $\tilde{K}(p)$ will give us $\frac{1}{p\tilde{K}(p)}$ and we can do the inverse transform by inspection.
Example Generalized Abel's equation has $K(\xi)=\xi^{-\mu}$

$$f(x)=\textstyle\int_0^x(x-y)^{-\mu}u(y)dy\quad 0<\mu<1 \tag{12}$$

Knowing the formula for $\tilde{K}(p)$

$$\tilde{K}(p)=\int_0^\infty \xi^{-\mu}e^{-p\xi}d\xi=p^{-(1-\mu)}\Gamma(1-\mu) \tag{13}$$

we can solve (10) with a Laplace transform

$$u(x)=\mathcal{L}^{-1}\left(\frac{\tilde{f}(p)}{\tilde{K}(p)}\right)=\frac{1}{\Gamma(1-\mu)}\mathcal{L}^{-1}\left(p^{1-\mu}\tilde{f}(p)\right) \tag{14}$$

subtituting $\mu \to 2-\mu$ in (13) we have

$$\int_0^\infty \xi^{\mu-2} e^{-p\xi} d\xi = p^{1-\mu}\Gamma(\mu-1) \Rightarrow$$
$$p^{1-\mu} = \frac{1}{\Gamma(\mu-1)}\widetilde{\xi^{\mu-2}}(p) \tag{15}$$

Thus we can do (14) by inspection

$$\begin{aligned} u(x) =& \frac{1}{\Gamma(1-\mu)\Gamma(\mu-1)}\mathcal{L}^{-1}\left(\widetilde{\xi^{\mu-2}}(p)\tilde{f}(p)\right) \\ =& \frac{1}{\Gamma(1-\mu)\Gamma(\mu-1)}\int_0^x (x-y)^{\mu-2} f(y)dy \end{aligned} \tag{16}$$

Interestingly, using the fact that $\tilde{f'} = p\tilde{f}$ we can write the same solution as

$$u(x) = \frac{1}{\Gamma(1-\mu)}\mathcal{L}^{-1}\left(p^{-\mu}\widetilde{f'}(p)\right) \tag{17}$$

subtituting $\mu \to 1-\mu$ in (13) we have

$$\int_0^\infty \xi^{\mu-1} e^{-p\xi} d\xi = p^{-\mu}\Gamma(\mu) \Rightarrow$$
$$p^{-\mu} = \frac{1}{\Gamma(\mu)}\widetilde{\xi^{\mu-1}}(p) \tag{18}$$

and

$$\begin{aligned} u(x) =& \frac{1}{\Gamma(1-\mu)\Gamma(\mu)}\mathcal{L}^{-1}\left(\widetilde{\xi^{\mu-1}}(p)\widetilde{f'}(p)\right) \\ =& \frac{1}{\Gamma(1-\mu)\Gamma(\mu)}\int_0^x (x-y)^{\mu-1} f'(y)dy \end{aligned} \tag{19}$$