Heat Equation

The stereotypical form of a 1D Heat/Diffusion equation is

$$\frac{\partial \phi(x,t)}{\partial t} - \kappa \frac{\partial^2 \phi(x,t)}{\partial x^2} = q(x,t) \tag{1}$$

with some known initial heat distribution at time $t_o = \eta$, $\phi(x,\tau)$.

Homogeneous Solution

The homogeneous equation

$$\frac{\partial \phi_o(x,t)}{\partial t} = \kappa \frac{\partial^2 \phi_o(x,t)}{\partial x^2} \tag{2}$$

can be solved quite easily with a spatial Fourier transform. Suppose

$$\phi_o(x,t) = \int \frac{dk}{2\pi} e^{ikx} \tilde{\phi}_o(k,t) \tag{3}$$

plugging into (2) to get

$$\begin{aligned} \int \frac{dk}{2\pi} e^{ikx} \left\{ \frac{\partial \tilde{\phi}_o}{\partial t} \right\} = \int \frac{dk}{2\pi} e^{ikx} \left\{ -\kappa k^2 \tilde{\phi}_o \right\} \Rightarrow \\ \frac{\partial \tilde{\phi}_o}{\partial t} = -\kappa k^2 \tilde{\phi}_o \end{aligned} \tag{4}$$

separate and integrate from τ to t we get $\tilde{\phi}_o$

$$\begin{aligned} \ln \tilde{\phi}_o(k,t')\big|_\tau^t = -\kappa k^2 (t-\tau) \Rightarrow \\ \tilde{\phi}_o(k,t) = \tilde{\phi}_o(k,\tau) e^{-\kappa k^2 (t-\tau)} \end{aligned} \tag{5}$$

plugging the result back into (3) to get the solution

$$\phi_o(x,t) = \int \frac{dk}{2\pi} \tilde{\phi}_o(k,\tau) e^{ikx - \kappa k^2 (t-\tau)} \tag{6}$$

Writing $\tilde{\phi}_o(k,\tau)$ in real space $\tilde{\phi}_o(k,\tau) = \int d\chi e^{-ik\chi} \phi_o(\chi,\tau)$ and we have

$$\boxed{\phi_o(x,t) = \int d\chi \left\{ \int \frac{dk}{2\pi} e^{ik(x-\chi) - \kappa k^2 (t-\tau)} \right\} \phi_o(\chi,\tau)} \tag{7}$$

Here it becomes natural to define the *heat kernel*

$$K(x-\chi, t-\tau) = \int \frac{dk}{2\pi} e^{ik(x-\chi)-\kappa k^2(t-\tau)}$$
$$= \frac{1}{\sqrt{4\pi\kappa(t-\tau)}} e^{-\frac{(x-\chi)^2}{4\kappa(t-\tau)}} \tag{8}$$

which turns out to be the causal Green's function for this problem as we shall soon see. The heat kernel describes the evolution of a unit blob of heat initially concentrated at $x = \chi$, $t = \tau$.

Green's Function

With homogeneous boundary conditions $\phi(x, 0) = 0$ the Green's function for $\hat{L} = \frac{\partial}{\partial t} - \kappa \frac{\partial^2}{\partial x^2}$ takes the causal form

$$G(x, t; \xi, \tau) = \begin{cases} 0 & t < \tau \\ \phi_o & t > \tau \end{cases} \tag{9}$$

The jump condition obtained by integrating

$$\hat{L} G(x, t; \xi, \tau) = \delta(x - \xi)\delta(t - \tau) \tag{10}$$

around $t \in (\tau - \epsilon, \tau + \epsilon)$ says

$$\lim_{t \to \tau^+} G(x, t; \xi, \tau) = \delta(x - \xi) \tag{11}$$

In other words, the Green's function starts out as $\delta(x - \xi)$ at $t = \tau$ and then evolves according to the homogeneous heat equation. Therefore, the Green's function at a later time can be calculated using the heat kernel.

$$G(x, t; \xi, \tau) = \int d\chi K(x - \chi, t - \tau)\delta(\chi - \xi) = K(x - \xi, t - \tau) \tag{12}$$

as promised, the heat kernel is indeed the causal Green's function for this problem. For completeness's sake, the final solution to the inhomogeneous problem

$$\boxed{\phi(x, t) = \int d\tau \int d\xi \left\{ \int \frac{dk}{2\pi} e^{ik(x-\xi)-\kappa k^2(t-\tau)} \right\} q(\xi, \tau)} \tag{13}$$