Green's Function

Motivation

Green function enables us to construct solutions to inhomogeneous equations

$$\hat{L}u(x) = f(x), \quad \mathscr{B} = \{u \in L^2[a,b] | u(a) = u_a, u(b) = u_b\} \tag{1}$$

from homogeneous solutions u_o with $\hat{L}u_o(x) = 0$. To appreciate the usefulness of Green's function, notice that for "nice enough" problems, if

$$\hat{L}G(x,\xi) = \delta(x-\xi) \tag{2}$$

then

$$u(x) = \int_a^b G(x,y)f(y)dy \tag{3}$$

is a solution to (1), since

$$\hat{L}u(x) = \int_a^b \hat{L}G(x,y)f(y)dy = \int_a^b \delta(x-y)f(y)dy = f(x) \tag{4}$$

In essence, we now only has to solve (2) instead of (1). Given that we know the form of the homogeneous solutions u_o, we can guess

$$G(x,\xi) = \begin{cases} u_{oL}(x) & x < \xi \\ u_{oR}(x) & x \geq \xi \end{cases} \tag{5}$$

in general, where $\hat{L}u_{oL/R} = 0$ and u_{oL}, u_{oR} satisfy the left/right boundary conditions respectively. Now, by integrating (2) around $(\xi-\epsilon, \xi+\epsilon)$, depending on what $\hat{L}$ looks like, we get a jump and various continuity conditions.

$$\begin{cases} \int_-^+ \hat{L}G(x,\xi)dx = 1 \\ \int_-^+ \int_a^x \hat{L}G(x_1,\xi)dx_1 dx = 0 \\ \int_-^+ \int_a^x \int_a^{x_2} \hat{L}G(x_1,\xi)dx_1 dx_2 = 0 \\ \vdots \end{cases} \tag{6}$$

Example $\hat{L} = -\frac{d^2}{dx^2}$ on $[0,1]$, $\hat{L}y(x) = f(x)$ and $y(0) = y(1) = 0$
Let $y_{oL} = ax$, $y_{oR} = b(1-x)$ then

$$G(x,\xi) = \begin{cases} y_{oL}(x) = ax & x < \xi \\ y_{oR}(x) = b(1-x) & x \geq \xi \end{cases} \tag{7}$$

the jump and continuity conditions are

$$\begin{cases} -G'(x,\xi)|_-^+ = 1 \\ -G(x,\xi)|_-^+ = 0 \end{cases} \Rightarrow \begin{cases} a+b=1 \\ a\xi = b(1-\xi) \end{cases} \Rightarrow \begin{cases} a = 1-\xi \\ b=\xi \end{cases} \tag{8}$$

thus the Green's function

$$G(x,\xi) = \begin{cases} x(1-\xi) & x < \xi \\ \xi(1-x) & x \geq \xi \end{cases} \tag{9}$$

and the solution

$$y(x) = \int\limits_0^x \xi(1-x)f(\xi)d\xi + \int\limits_x^1 x(1-\xi)f(\xi)d\xi \tag{10}$$

Adjoint Green's Function

To see what is meant by "nice enough", let's first look at the *adjoint Green's function* satisfying

$$\hat{L}^\dagger G^\dagger(x,\xi) = \delta(x-\xi) \tag{11}$$

If $\hat{L}$ on $\mathscr{D}$ is self-adjoint with $\hat{L}^\dagger$ on $\mathscr{D}^\dagger$, Lagrange's identity says

$$\int_a^b u^*(x)\hat{L}v(x) - (\hat{L}^\dagger u(x))^* v(x)dx = 0 \tag{12}$$

Plugging in

$$\begin{cases} u(x) = G^\dagger(x,\xi) \\ v(x) = G(x,\xi') \end{cases} \tag{13}$$

and we find

$$\int_a^b (G^\dagger(x,\xi))^*(x)\delta(x-\xi') - (\delta(x-\xi))^* G(x,\xi')dx = 0 \Rightarrow$$
$$\boxed{(G^\dagger(\xi',\xi))^* = G(\xi,\xi')} \tag{14}$$

Now consider the general boundary value problem (1) where $\mathscr{B} \neq \mathscr{D}$. For the set of functions that may be a solution $\mathscr{B}$, the operator $\hat{L}$ on $\mathscr{B}$ may no long be self-adjoint. Thus Lagrange's identity reads[1]

$$\int_a^b \mu^*(y)\hat{L}v(y) - (\hat{L}^\dagger \mu(y))^* v(y)dy = Q|_a^b \tag{15}$$

[1] I'm using μ instead of u to not confuse it with the u in (1)

Suppose we have already found $G, G^\dagger \in \mathscr{D}$ and let

$$\begin{cases} \mu(y) = G^\dagger(y,x) \\ v(y) = u(y) \end{cases} \tag{16}$$

then (15) becomes

$$\int_a^b (G^\dagger(y,x))^* f(y) - \delta(y,x)u(y)dy = Q|_a^b \Rightarrow$$

$$u(x) = \int_a^b G(x,y)f(y)dy - Q|_a^b \tag{17}$$

Comparing equations (3) and (17) we finally see that "nice enough" means the boundary conditions of the problem $\mathscr{B}$ must make $Q|_a^b$ vanish.