Generating Function and Normalization

Generating functions are often nice and integrable and give us a way to calculate the normalization of the special functions they generate.

Hermite Polynomial

The generating function for Hermite polynomials

$$G_t(x) = e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} \tag{1}$$

Knowing the inner product has weight $w(x) = e^{-x^2}$ on $x \in (-\infty, \infty)$

$$\langle G_s, G_t \rangle = \int_{-\infty}^{\infty} e^{-x^2} e^{2xs-s^2} e^{2xt-t^2} dx = \sqrt{\pi} e^{2st} \tag{2}$$

Writing (2) in series expansion

$$\begin{aligned}\langle G_s, G_t \rangle =& \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\int_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)dx\right)\cdot\frac{s^m t^n}{m!n!} \\ =& \sum_{n=0}^{\infty}\sqrt{\pi}2^n n!\cdot\frac{s^n t^n}{n!n!} \end{aligned}\tag{3}$$

Observe

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)dx = \sqrt{\pi}2^n n!\delta_{mn} \tag{4}$$

Legendre Polynomial

The generating function for Legendre polynomials

$$G_z(x) = \frac{1}{\sqrt{1-2zx+z^2}} = \sum_{l=0}^{\infty} z^l P_l(x) \quad |z| < 1 \tag{5}$$

Knowing the inner product has weight $w(x) = 1$ on $x \in (-1, 1)$

$$\begin{aligned}\langle G_y, G_z \rangle =& \int_{-1}^{1} \frac{1}{\sqrt{1-2yx+y^2}}\frac{1}{\sqrt{1-2zx+z^2}} \\ =& \frac{1}{\sqrt{yz}}\ln\frac{\sqrt{yz}+1}{\sqrt{yz}-1}\end{aligned}\tag{6}$$

Writing (6) in series expansion [1]

$$
\begin{aligned}
\langle G_y, G_z \rangle &= \sum_{l=0}^{\infty} \left(\int_{-1}^{1} P_m(x) P_l(x) dx \right) y^m z^l \\
&= \sum_{l=0}^{\infty} \frac{2}{2l+1} y^l z^l
\end{aligned} \tag{7}
$$

Observe

$$
\int_{-1}^{1} P_m(x) P_l(x) dx = \frac{2}{2l+1} \delta_{ml} \tag{8}
$$

[1] Log expansion http://www.math.com/tables/expansion/log.htm