Distributional Derivative

In this section, we shall find a place for ”nasty” ”functions” (distributions) such as the Dirac delta function to live (dual space of function space) and find a general rule for differentiation in this space (distributional/weak derivative).

Dual Space

A function takes elements in its domain to its range. Consider a space of functions \( V = \{ f : D \to \mathcal{R} | D, \mathcal{R} \in \mathcal{F} \} \) where \( \mathcal{F} \) is the field over which the functions in \( V \) are defined, then it’s dual space \( V^* \) is the space of linear and continuous functionals that take elements of \( V \) and map them back to \( \mathcal{F} \) i.e. \( V^* = \{ F : V \to \mathcal{F} | \text{linear and continuous} \} \).

For example, the Hilbert space is its own dual by the Riesz—Fréchet theorem: any continuous linear map \( F : L^2 \to \mathbb{R} \) can be written as \( F[f] = \langle l, f \rangle \) for some \( l \in L^2 \).

Test Function Space

Test functions are ”nice” functions satisfying

1. Smooth
2. Vanishes at infinity

Examples of such spaces include the Schwartz space \( \mathcal{S}(\mathbb{R}) \) and the space of smooth functions with compact support \( C_0^\infty \). The space of relevant test functions is generally referred to as \( \mathcal{T} \), in other words, depending on the context \( \mathcal{T} = \mathcal{S}(\mathbb{R}), C_0^\infty \) etc.

Distribution

Distributions are elements of the dual space \( V^* \) of a function space \( V \). In general, elements of \( V \) need not be test functions, however, more often than not they are because the ”nicer” \( V \) is the ”nastier” the functions in \( V^* \) can be. For example, \( \langle \delta(x) \rangle \) is not an element of the dual space of the Hilbert space (which is just the Hilbert space itself), but it can be accommodated in \( \mathcal{T}^* \). That is for \( \phi \in \mathcal{T} \)

\[
\delta_{x-a}[\phi] \equiv \phi(a)
\] (1)
We still use the conventional notation
\[ \int_{-\infty}^{\infty} \delta(x - a) \phi(x) \, dx = \phi(a) \]  
(2)
because it suggests correct results such as
\[ \int_{-\infty}^{\infty} \delta(ax - b) \phi(x) \, dx = \frac{1}{|a|} \phi(b/a) \]  
(3)
The derivatives of \( \delta(x) \) can then be defined by paring them with derivative functions in \( T \)
\[ \delta'_{x-a}[\phi] \equiv -\phi'(a) = -\delta_{x-a}[\phi'] \]  
(4)
suggested by the conventional notation
\[ \int_{-\infty}^{\infty} \phi(x) \frac{d}{dx} \delta(x - a) \, dx = -\int_{-\infty}^{\infty} \phi'(x) \delta(x - a) \, dx = -\phi'(a) \]  
(5)
The form of equation (4) can serve as the definition for a general derivative of a distribution, that is for \( d \in T^* \) and \( \phi \in T \)
\[ d'[\phi] \equiv -d[\phi'] \]  
(6)
This is known as a distributional derivative or a weak derivative