Distributional Derivative

In this section, we shall find a place for "nasty" "functions"(distributions) such as the Dirac delta function to live (dual space of function space) and find a general rule for differentiation in this space (distributional/weak derivative).

Dual Space

A function takes elements in its domain to its range. Consider a space of functions $V = \{f : \mathcal{D} \to \mathcal{R} | \mathcal{D}, \mathcal{R} \in \mathcal{F}\}$ where $\mathcal{F}$ is the field over which the functions in V are defined, then it's dual space V^* is the space of linear and continuous functionals that take elements of V and map them back to $\mathcal{F}$ i.e. $V^* = \{F : V \to \mathcal{F} | \text{linear and continuous}\}$.

For example, the *Hilbert space is its own dual* by the Riesz—Fréchet theorem: any continuous linear map $F : L^2 \to \mathbb{R}$ can be written as $F[f] = \langle l, f \rangle$ for some $l \in L^2$

Test Function Space

Test functions are "nice" functions satisfying

1. Smooth
2. Vanishes at infinity

Examples of such spaces include the *Schwartz space* $\mathcal{S}(\mathbb{R})$ and the space of smooth functions with compact support C_0^∞. The space of relevant test functions is generally referred to as $\mathcal{T}$, in other words, depending on the context $\mathcal{T} = \mathcal{S}(\mathbb{R}), C_0^\infty$ etc.

Distribution

Distributions are elements of the dual space V^* of a function space V. In general, elements of V need not be test functions, however, more often than not they are because the "nicer" V is the "nastier" the functions in V^* can be. For example, $\langle \delta(x) |$ is not an element of the dual space of the Hilbert space (which is just the Hilbert space itself), but it can be accommodated in $\mathcal{T}^*$. That is for $\phi \in \mathcal{T}$

$$\delta_{x-a}[\phi] \equiv \phi(a) \tag{1}$$

We still use the conventional notation

$$\int_{-\infty}^{\infty} \delta(x-a)\phi(x)dx = \phi(a) \tag{2}$$

because it suggests correct results such as

$$\int_{-\infty}^{\infty} \delta(ax-b)\phi(x)dx = \frac{1}{|a|}\phi(b/a) \tag{3}$$

The derivatives of $\delta(x)$ can then be defined by paring them with derivative functions in $\mathcal{T}$

$$\delta'_{x-a}[\phi] \equiv -\phi'(a) = -\delta_{x-a}[\phi'] \tag{4}$$

suggested by the conventional notation

$$\int_{-\infty}^{\infty} \phi(x)\frac{d}{dx}\delta(x-a)dx = -\int_{-\infty}^{\infty} \phi'(x)\delta(x-a)dx = -\phi'(a) \tag{5}$$

The form of equation (4) can serve as the definition for a general derivative of a distribution, that is for $d \in \mathcal{T}^*$ and $\phi \in \mathcal{T}$

$$\boxed{d'[\phi] \equiv -d[\phi']} \tag{6}$$

This is known as a *distributional derivative* or a *weak derivative*