# Density of States

*Density of states* $D(\epsilon)$ is a basic quantum mechanics function that measures the density of eigenstates at a given energy level $\epsilon$. It is mostly easily calculated when the system is large and its dispersion relation is spherically symmetric with respect to the quantum numbers. That is if a state is labeled by $\vec{n} = (n_1, n_2, n_3, \cdots)$, then the energy of a state $\epsilon(\vec{n})$ only depends on the norm $n = \sqrt{\sum_i n_i^2}$. For a big system where spherical symmetry is available, the total number of states may be approximated as an energy integral.

$$\begin{cases} N_{1D} \approx \int dn = \int \frac{dn}{d\epsilon} d\epsilon \\ N_{2D} \approx \iint dn_x dn_y = \int_0^{2\pi} \int n dn d\phi = \int 2\pi n dn = \int 2\pi n \frac{dn}{d\epsilon} d\epsilon \\ N_{3D} \approx \iiint dn_x dn_y dn_z = \int_0^{2\pi} \int_0^{\pi} \int n^2 \sin\theta n dn d\theta d\pi \\ = \int 4\pi n^2 dn = \int 4\pi n^2 \frac{dn}{d\epsilon} d\epsilon \end{cases} \tag{1}$$

By the definition of density of states, we can also write the total number of eigenstates as

$$N_{1,2,3D} = \int D(\epsilon) d\epsilon \tag{2}$$

Comparing (1) with (2), we can read off the density of states

$$\boxed{\begin{cases} D_{1D}(\epsilon) = \frac{dn}{d\epsilon} \\ D_{2D}(\epsilon) = 2\pi n \frac{dn}{d\epsilon} \\ D_{3D}(\epsilon) = 4\pi n^2 \frac{dn}{d\epsilon} \end{cases}} \tag{3}$$

### Free particle

Schrödinger's equation for a free particle in 1D

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) = E\psi(x) \tag{4}$$

with periodic condition on $x \in [-\frac{L}{2}, \frac{L}{2}]$, the eigenstates are

$$\psi_n(x) = \frac{1}{\sqrt{L}} e^{-i\frac{2\pi n x}{L}} \tag{5}$$

and the eigenvalues give

$$\epsilon = \frac{\hbar^2}{2m} \frac{4\pi^2 n^2}{L^2} \Rightarrow n = L / \frac{2\pi\hbar}{\sqrt{2m\epsilon}} \tag{6}$$

(6) holds for higher dimensions, therefore the density of states for free particles in 1,2,3D are

$$\begin{cases} D_{1D}(\epsilon) = L \cdot \frac{\sqrt{2m}}{4\pi\hbar} \cdot \frac{1}{\sqrt{\epsilon}} \\ D_{2D}(\epsilon) = L^2 \cdot \frac{2m}{4\pi\hbar^2} \\ D_{3D}(\epsilon) = L^3 \cdot \frac{(2m)^{3/2}}{4\pi^2\hbar^3} \cdot \sqrt{\epsilon} \end{cases} \tag{7}$$

Stare at (6) long enough and we'll see that we can interpret

$$\ell_Q = \frac{h}{\sqrt{2m\epsilon}} \tag{8}$$

as the linear size of an eigenstate in real space. This kind of thinking will become useful again during the discussion of Landau levels.