Damped Harmonic Oscillator

The damped harmonic oscillator problem is an excellent place to practice using *Reduction of Order* and *Green's function* to elegantly solve an ODE.

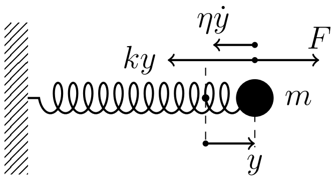

Figure 1: Damped Harmonic Oscillator

Starting with $F = ma$, we have the elementary form

$$F(t) - ky(t) - \eta\dot{y}(t) = m\ddot{y}(t) \tag{1}$$

where k is the spring constant and η is the laminar drag coefficient. For initial conditions, suppose the oscillator starts from rest and the force turns on at $t = 0$, that is $y(0) = 0$, $y'(0) = 0$.

Standard form

Using notations (γ, Ω) that will make sense later, we can rewrite the above elementary form into the standard form that we have been seeing.

$$p_0y'' + p_1y' + p_2y = y'' + 2\gamma y' + (\Omega^2 + \gamma^2)y = F \tag{2}$$

Sturm—Liouville form

Using reduction of order, we can recast (2) into Sturm-Liouville form

$$\tilde{y}'' + \Omega\tilde{y} = F \tag{3}$$

where

$$\begin{cases} y = \omega\tilde{y} \\ \omega = e^{-\gamma(t-a)} \end{cases} \tag{4}$$

Homogeneous Solution

We can solve $\tilde{y}'' + \Omega\tilde{y} = 0$ by inspection

$$\tilde{y}_o = A\sin(\Omega t + \phi) \tag{5}$$

and would thus solve $y'' + 2\gamma y' + (\Omega^2 + \gamma^2)y = 0$ by (4)

$$y_o = (Ae^{\gamma a})e^{-\gamma t}\sin(\Omega t + \phi) \tag{6}$$

Green's Function

With the homogeneous solution obtained, we move on to solve the inhomogeneous problem. Since the only $y_o(t)$ that satiesfies $y_o(0) = 0$ and $y_o'(0) = 0$ is $y_o(t) = 0$, the Green's function takes the causal form

$$\begin{aligned} G(t,\tau) = &\begin{cases} 0 & t < \tau \\ y_o & t > \tau \end{cases} \\ =&\Theta(t-\tau)\cdot Be^{-\gamma t}\sin(\Omega t + \phi) \end{aligned} \tag{7}$$

the jump and continuity conditions obtained from $\hat{L}G = \delta(t-\tau)$ say

$$\begin{cases} \partial_t G(t,\tau)|_{\tau-\epsilon}^{\tau+\epsilon} = 1 \\ G(t,\tau)|_{\tau-\epsilon}^{\tau+\epsilon} = 0 \end{cases} \tag{8}$$

continuity immediately gives $\phi = -\Omega\tau$, and then jump becomes

$$\begin{aligned} -\gamma Be^{-\gamma\tau}\sin\Omega(\tau-\tau) + \Omega Be^{-\gamma\tau}\cos\Omega(\tau-\tau) = 1 \Rightarrow \\ B = \frac{1}{\Omega}\cdot e^{\gamma\tau} \end{aligned} \tag{9}$$

Therefore, the Green's function

$$G(t,\tau) = \Theta(t-\tau)\cdot\frac{1}{\Omega}e^{-\gamma(t-\tau)}\sin\Omega(t-\tau) \tag{10}$$

and the final solution

$$\boxed{y(t) = \int_0^t d\tau\frac{1}{\Omega}e^{-\gamma(t-\tau)}\sin\Omega(t-\tau)F(\tau)} \tag{11}$$

Lo and behold, $\gamma = \frac{\eta}{2m}$ is the exponential decay factor due to drag and $\Omega = \sqrt{\frac{k}{m} - (\frac{\eta}{2m})^2}$ is the spring-mass system's oscillation frequency modified by drag. We now have an intuitive sense of what the Green function is (at least in this case). $G(t,\tau)$ is the response of the system to a kick at $t = \tau$, as expected the response $\frac{1}{\Omega}e^{-\gamma(t-\tau)}\sin\Omega(t-\tau)$ is a damped oscillation that dies over time. When a driving force is present, different responses triggered by the driving force super-impose on each other, forming the final solution.