Damped Harmonic Oscillator

The damped harmonic oscillator problem is an excellent place to practice using *Reduction of Order* and *Green’s function* to elegantly solve an ODE.

Figure 1: Damped Harmonic Oscillator

Starting with $F = ma$, we have the elementary form

$$F(t) - ky(t) - \eta \dot{y}(t) = m\ddot{y}(t)$$  \hspace{1cm} (1)

where $k$ is the spring constant and $\eta$ is the laminar drag coefficient. For initial conditions, suppose the oscillator starts from rest and the force turns on at $t = 0$, that is $y(0) = 0$, $y'(0) = 0$.

**Standard form**

Using notations $(\gamma, \Omega)$ that will make sense later, we can rewrite the above elementary form into the standard form that we have been seeing.

$$p_0y'' + p_1y' + p_2y = y'' + 2\gamma y' + (\Omega^2 + \gamma^2)y = F$$  \hspace{1cm} (2)

**Sturm—Liouville form**

Using reduction of order, we can recast (2) into Sturm-Liouville form

$$\tilde{y}'' + \Omega \tilde{y} = F$$  \hspace{1cm} (3)

where

$$\begin{cases} y = \omega \tilde{y} \\ \omega = e^{-\gamma(t-a)} \end{cases}$$  \hspace{1cm} (4)

**Homogeneous Solution**

We can solve $\tilde{y}'' + \Omega \tilde{y} = 0$ by inspection

$$\tilde{y}_o = A \sin(\Omega t + \phi)$$  \hspace{1cm} (5)

and would thus solve $y'' + 2\gamma y' + (\Omega^2 + \gamma^2)y = 0$ by (4)

$$y_o = (A e^{\gamma a}) e^{-\gamma t} \sin(\Omega t + \phi)$$  \hspace{1cm} (6)
Green’s Function

With the homogeneous solution obtained, we move on to solve the inhomogeneous problem. Since the only $y_o(t)$ that satisfies $y_o(0) = 0$ and $y_o'(0) = 0$ is $y_o(t) = 0$, the Green’s function takes the causal form

$$G(t, \tau) = \begin{cases} 0 & t < \tau \\ y_o & t > \tau \end{cases} = \Theta(t - \tau) \cdot Be^{-\gamma t} \sin(\Omega t + \phi)$$

(7)

the jump and continuity conditions obtained from $\hat{L}G = \delta(t - \tau)$ say

$$\begin{cases} \partial_t G(t, \tau) |_{\tau + \epsilon}^{\tau - \epsilon} = 1 \\ G(t, \tau) |_{\tau + \epsilon}^{\tau - \epsilon} = 0 \end{cases}$$

(8)

continuity immediately gives $\phi = -\Omega \tau$, and then jump becomes

$$-\gamma Be^{-\gamma \tau} \sin \Omega(\tau - \tau) + \Omega Be^{-\gamma \tau} \cos \Omega(\tau - \tau) = 1 \Rightarrow B = \frac{1}{\Omega} e^{\gamma \tau}$$

(9)

Therefore, the Green’s function

$$G(t, \tau) = \Theta(t - \tau) \cdot \frac{1}{\Omega} e^{-\gamma(t-\tau)} \sin \Omega(t - \tau)$$

(10)

and the final solution

$$y(t) = \int_0^t \frac{1}{\Omega} e^{-\gamma(t-\tau)} \sin \Omega(t - \tau) F(\tau)$$

(11)

Lo and behold, $\gamma = \frac{\eta}{2m}$ is the exponential decay factor due to drag and $\Omega = \sqrt{\frac{k}{m} - \left(\frac{\eta}{2m}\right)^2}$ is the spring-mass system’s oscillation frequency modified by drag. We now have an intuitive sense of what the Green function is (at least in this case). $G(t, \tau)$ is the response of the system to a kick at $t = \tau$, as expected the response $\frac{1}{\Omega} e^{-\gamma(t-\tau)} \sin \Omega(t - \tau)$ is a damped oscillation that dies over time. When a driving force is present, different responses triggered by the driving force super-impose on each other, forming the final solution.