Completeness

Definition

A complete basis for a given space can expand any element of that space. For L^2, it turns out as long as a basis can expand the delta function, it can expand any function in L^2. Thus the *completeness relation* states:
For a set of basis functions $\{\phi_\lambda\}$, if $\exists$ coefficients $\tilde{f}(\lambda)$ such that

$$\delta(x - x') = \int d\lambda \tilde{f}(\lambda)\phi_\lambda(x) \tag{1}$$

then $\{\phi_\lambda\}$ is complete.

Orthogonality and Completeness

If the set of basis functions are orthogonal, then we can derive an alternative statement of completeness. Suppose $\phi_\lambda \perp \phi_\mu$ (not necessarily normalized)

$$\int \phi_\lambda(x)\phi_\mu(x)dx = A_{\lambda,\mu}\delta(\lambda - \mu) \tag{2}$$

Then we can calculate $\tilde{f}$ using Fourier's trick

$$\begin{aligned}\int \phi_\mu(x)\delta(x - x')dx = \int \phi_\mu(x) \int d\lambda \tilde{f}(\lambda)\phi_\lambda(x)dx \Rightarrow \\ \phi_\mu(x') = A_{\mu,\mu}\tilde{f}(\mu) \Rightarrow \\ \tilde{f}(\mu) = \phi_\mu(x')/A_{\mu,\mu}\end{aligned} \tag{3}$$

Therefore, the completeness relation (1) can be rewritten as

$$\boxed{\delta(x - x') = \int \frac{d\lambda}{A_{\lambda,\lambda}}\phi_\lambda(x')\phi_\lambda(x)} \tag{4}$$