Bessel's Equation

Bessel's equation can be written in the form

$$\hat{L}_n y = -y'' - \frac{1}{x}y' + \frac{n^2}{x^2}y = k^2 y \tag{1}$$

the eigenfunctions J_n are the Bessel functions satisfying

$$\hat{L}_n J_n = k^2 J_n \tag{2}$$

Factorization

using reduction of order, the LHS can be put into Sturm-Liouville form

$$-\frac{1}{x}\frac{d}{dx}x\frac{d}{dx}y + \frac{n^2}{x^2}y = k^2 y \tag{3}$$

and we can read off the weight that makes it formally self-adjoint $w = x$. We also notice that the operator in (1) can be factored in two ways

$$\begin{aligned}\hat{L}_n &= -\frac{d^2}{dx^2} - \frac{1}{x}\frac{d}{dx} + \frac{n^2}{x^2} \\ &= \left(\frac{d}{dx} + \frac{n+1}{x}\right)\left(-\frac{d}{dx} + \frac{n}{x}\right) \\ &= \left(-\frac{d}{dx} + \frac{n-1}{x}\right)\left(\frac{d}{dx} + \frac{n}{x}\right)\end{aligned} \tag{4}$$

Defining the raising operator

$$\boxed{\hat{A}_n = -\frac{d}{dx} + \frac{n}{x}} \tag{5}$$

we see that

$$\frac{d}{dx} + \frac{n+1}{x} \tag{6}$$

happens to be its ajoint $\hat{A}_n^\dagger$ with respect to the weight $w = x$. Thus

$$\boxed{\hat{L}_n = \hat{A}_n^\dagger \hat{A}_n = \hat{A}_{n-1}\hat{A}_{n-1}^\dagger} \tag{7}$$

This shouldn't be too surprising in hind sight. Since $\hat{L}_n$ is formally self-adjoint, we'd better be able to find some operator $\hat{A}$ such that $\hat{L}_n = \hat{A}^\dagger \hat{A}$.

Bessel Identities

The factorization property of $\hat{L}_n$ results in various identities for J_n. Consider the effect of $\hat{A}_n$ on J_n

$$\hat{A}_n J_n = \hat{A}_n\left(\frac{1}{k^2}\hat{L}_n J_n\right) = \frac{1}{k^2}\hat{A}_n\hat{A}_n^\dagger\hat{A}_n J_n = \frac{1}{k^2}\hat{A}_{n+1}^\dagger\hat{A}_{n+1}(\hat{A}_n J_n) \Rightarrow$$
$$k^2(\hat{A}_n J_n) = \hat{L}_{n+1}(\hat{A}_n J_n) \tag{8}$$

Voila, $\hat{A}_n J_n = J_{n+1}$! Similarly $\hat{A}_{n-1}^\dagger J_n = J_{n-1}$. Explicitly

$$\begin{cases} (-\frac{d}{dx}+\frac{n}{x})J_n = J_{n+1} \\ (\frac{d}{dx}+\frac{n}{x})J_n = J_{n-1} \end{cases} \Rightarrow$$
$$\boxed{\begin{cases} \frac{2n}{x}J_n = J_{n+1}+J_{n-1} \\ 2J_n' = J_{n-1}-J_{n+1} \end{cases}} \tag{9}$$

Generating Function

The generating function for Bessel functions

$$G_z(x) = e^{\frac{1}{2}x(z-\frac{1}{z})} = \sum J_n(x)z^n \tag{10}$$

we can easily reproduce *Bessel Identities* with this generating function. First notice

$$\begin{cases} zG_z = \sum J_n z^{n+1} = \sum J_{n-1}z^n \\ \frac{1}{z}G_z = \sum J_n z^{n-1} = \sum J_{n+1}z^n \end{cases} \tag{11}$$

Taking the x derivative of both sides of (10)

$$\frac{1}{2}(z-\frac{1}{z})G_z = \sum J_n'(x)z^n \Rightarrow$$
$$\frac{1}{2}\sum(J_{n-1}-J_{n+1})z^n = \sum J_n' z^n \Rightarrow$$
$$J_{n-1}-J_{n+1} = 2J_n' \tag{12}$$

Taking the t derivative of both sides of (10)

$$\frac{1}{2}x\frac{1}{z}(z+\frac{1}{z})G_z = \sum nJ_n(x)z^{n-1} \Rightarrow$$
$$\frac{1}{2}x\sum(J_{n-1}+J_{n+1})z^{n-1} = \sum nJ_n z^{n-1} \Rightarrow$$
$$J_{n-1}+J_{n+1} = \frac{2n}{x}J_n \tag{13}$$