Burger's Shock

In the fluid's frame of reference, Riemann's equations reduce to

$$\partial_t u + u\partial_x u = 0 \tag{1}$$

multiplying by u^{n-1} to get the form

$$\partial_t(\frac{u^n}{n}) + \partial_x(\frac{u^{n+1}}{n+1}) = 0 \tag{2}$$

which contains a conservation law that can be revealed by integration

$$\frac{1}{n}\partial_t(\int_{-\infty}^{\infty} u^n dx) = -[\frac{u^{n+1}}{n+1}]_{-\infty}^{\infty} = 0 \Rightarrow$$

$$\partial_t(\int_{-\infty}^{\infty} u^n dx) = 0 \tag{3}$$

Notice the RHS vanished since the wave u vanishes at infinity. This conservation law dictates the velocity of any *shock* (discontinuity) the wave might have. Suppose the wave is discontinuous at $x = X(t)$

$$\partial_t(\int_{-\infty}^{X(t)} u^n dx + \int_{X(t)}^{\infty} u^n dx) = 0 \Rightarrow$$

$$\dot{X}(u^n|_{-\infty} - u^n|_{\infty}) + \int_{-\infty}^{X(t)} \partial_t u^n dx + \int_{X(t)}^{\infty} \partial_t u^n dx = 0 \Rightarrow$$

$$(u_L^n - u_R^n)\dot{X} = \frac{n}{n+1}(\int_{-\infty}^{X(t)} \partial_x u^{n+1} dx + \int_{X(t)}^{\infty} \partial_x u^{n+1} dx) \Rightarrow$$

$$(u_L^n - u_R^n)\dot{X} = \frac{n}{n+1}(u_L^{n+1} - u_R^{n+1}) \Rightarrow$$

$$\dot{X} = \frac{n}{n+1}\frac{u_L^{n+1} - u_R^{n+1}}{u_L^n - u_R^n} \tag{4}$$

Berger's equation only admits one such conservation law

$$(\partial_t + u\partial_x)u = \nu\partial_x^2 u \Rightarrow$$

$$\partial_t u + \partial_x \left\{\frac{1}{2}u^2 - \nu\partial_x u\right\} = 0 \Rightarrow$$

$$\partial_t(\int u dx) = 0 \tag{5}$$

thus a Berger's shock moves with the average of the speeds of wave to its far left and right

$$\boxed{\dot{X} = \frac{1}{2}(u_L^n + u_R^n)} \tag{6}$$