Banach Space vs. Hilbert Space

A Banach space $L^p[a, b]$ is a complete normed vector space. A Hilbert space is essentially the same thing with its norm determined by an inner product. For example, $L^2[a, b]$ can be easily converted to a Hilbert space by defining the inner product $\langle f, g\rangle = \int f^*(x)g(x)dx$, which can be used as a norm $\|f\| = \langle f, f\rangle^{1/2}$ in this space. To full understand of these spaces, however, we need to introduce some important definitions.

Definitions

1. $C^\omega[a, b]$: Analytical Function
   A function whose Taylor series converges to the function itself, an interesting example of a non-analytic function is $y = e^{-1/x^2}$. Its Taylor expansion around $x = 0$ converges to 0 instead of the actual function.

2. $C_0^\infty[a, b]$: Smooth Function with Compact Support
   Important: $C_0^\omega[a, b] = \{0\}$, the only analytical function with compact support is the constant function 0

3. Point-Wise Convergence vs. Uniform Convergence

   (a) $f_n \to f$ point-wise if
   $\forall\epsilon > 0$ and $\forall x \in \mathcal{D}$, $\exists N_x \in \mathbb{N}$ s.t. $\|f_{N_x+}(x) - f(x)\| < \epsilon$

   (b) $f_n \to f$ uniformly if
   $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\|f_{N+}(x) - f(x)\| < \epsilon, \forall x \in \mathcal{D}$

4. $\|\ \|$: Norm

   (a) Positive Definite: $\|f\| \geq 0$ and $\|f\| = 0 \Leftrightarrow f = 0$

   (b) Cauchy-Schwarz: $\|f + g\| \leq \|f\| + \|g\|$

   (c) Linearly Homogeneous: $\|\lambda f\| = |\lambda|\|f\|$

   Examples:

   (a) $\|\ \|_\infty$ sup-norm
   $\|f\|_\infty = \sup_{x\in\mathcal{D}} |f(x)|$

   (b) $\|\ \|_p$ p-norm
   $\|f\|_p = (\int_{\mathcal{D}} |f(x)|^p dx)^{1/p}$
   Notice, indeed $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$