1D Sound Wave

The Original Problem

The simplest example of a non-linear wave is 1D sound wave. It obeys mass conservation and equation of motion

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \rho(\partial_t v + v\partial_x v) + \partial_x P = 0 \end{cases} \tag{1}$$

along with the equation of state $P(\rho)$.

Simplification

With *a priori* knowledge of the equation of state $P(\rho)$, we can unify all spatial and temporal dependent terms involving ρ and P under a new thermal dynamic variable. To achieve this, first play with partial derivatives and rewrite mass conservation using $\partial_\rho P$ to eliminate $\partial_\mu \rho$ ($\partial_\mu \rho \partial_\rho P = \partial_\mu P$)

$$\frac{1}{\partial_\rho P}(\partial_t \rho \partial_\rho P + v\partial_x \rho \partial_\rho P) + \rho \partial_x v = 0 \Rightarrow$$

$$\frac{1}{\rho}(\partial_t P + v\partial_x P) + \partial_\rho P \partial_x v = 0 \tag{2}$$

we can further eliminate ρ and $\partial_\mu P$ by defining a new thermal dynamic variable $\pi(P)$ such that $\partial_P \pi = \frac{1}{\rho}$

$$\begin{cases} \partial_t \pi + v\partial_x \pi + \partial_\rho P \partial_x v = 0 \\ \partial_t v + v\partial_x v + \partial_x \pi = 0 \end{cases} \tag{3}$$

Riemann's Solution

Historically, Riemann was the first to solve (1). Using his notations

$$\begin{cases} c^2 \equiv \partial_\rho P \\ \pi(P) \equiv \int_{P_o}^{P} \frac{1}{\rho c} dP \end{cases} \tag{4}$$

the simplified equations are slightly different

$$\begin{cases} \partial_t \pi + v\partial_x \pi + c\partial_x v = 0 \\ \partial_t v + v\partial_x v + c\partial_x \pi = 0 \end{cases} \tag{5}$$

adding and subtracting the two equations in (5)

$$\begin{cases} \left[\partial_t + (v+c)\partial_x\right](v+\pi) = 0 \\ \left[\partial_t + (v-c)\partial_x\right](v-\pi) = 0 \end{cases} \tag{6}$$

we may interpret the operators geometrically following Euler's characteristics. That is $v \pm \pi$ is constant along the characteristic defined by $\frac{dx}{dt} = v \pm c$.