Elementary Enumerative Combinatorics

Examples

Binomial Coefficients

$$3! \begin{pmatrix} 1 & 2 \end{pmatrix} \times \begin{pmatrix} 4 \end{pmatrix} 3 = \binom{n}{k}$$

$$2! \begin{pmatrix} 4 \end{pmatrix} \times \begin{pmatrix} 5 \end{pmatrix} 1 = \binom{n}{k}$$

Total # of all permutations when = \( \frac{n!}{k!(n-k)!} = \binom{n}{k} \)

Splitting into 3 subsets (of sizes 3, 2, 1)

$$\frac{6!}{3! \cdot 2! \cdot 1!} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = \frac{60}{6} = 60$$

$$\binom{n}{k} \text{ splits} = \frac{n!}{k! \cdot (n-k)!} = \binom{n}{k}$$
Symmetries

Elementary Enumerative Combinatorics

Examples

Binomial Coefficients

For example, if you want to split 13 elements into 6 groups of sizes 2, 2, 1, 4, 1, 3, the total # of such splits is \( \frac{13!}{2!2!1!4!1!3!} = 120,200 \)

\( \begin{align*}
\binom{n}{k_1, k_2, \ldots, k_{s-1}} &= \frac{n!}{k_1! k_2! \cdots k_{s-1}! (n-k_1-k_2-\cdots-k_{s-1})!} \\
(x+y+z)^6 &= \sum \binom{6}{k, k, k, 0, 0, 0} (x^3)(y^2)(z^1) + \cdots + 60 \times x^3 y^2 z^1 + \cdots + 1 \times x^{15} y^{15} z^{15} \\
&= \binom{15}{5}
\end{align*} \)
Anagrams

How many anagrams of ILLINI? 60

I L L N
I L L I N

BBEENNOOOR TTT

B²E²N⁴O⁴RT³

# of anagrams is 1,201,200

One of them is
To be or not to be?
Examples of computing probabilities in anagram spaces

Anagrams

a) \(G_{\text{ILLINE}}\)

\[ |A| = 12 \]

\[ A = A_1 U A_2 U A_3 U A_4 \]

\[ A_1 = \{ \text{ILL} \} \]

\[ A_2 = \{ \text{ILL} \} \]

\[ A_3 = \{ \text{ILL} \} \]

\[ A_4 = \{ \text{ILL} \} \]

where \(\text{ILL}\) consists of \(\ldots \text{ILL} \ldots\).

These three letters are disjoint and \(\text{ILL}\) is an anagram of \(11N\).

\[ |A| = \sum_{k=1}^{4} |A_k| \]

\[ |A| = 4 \cdot 12 \]

There are \(\frac{3!}{2!\cdot 1!\cdot 1!} = 3\) of them

\[ |G_{\text{MINI}}| = \]

\[ \text{ILL} \ldots \times 2 \]

\[ \text{ILL} \ldots \times 3 \]

\[ \times 3 \]

\[ \times 3 \times 3 \]

\[ \times 3 \]

\[ = \frac{4!}{2! \cdot 1! \cdot 1!} = 12 \]
Anagrams

Another version

b) How many anagrams will have a subsequence \( x^1 x^1 x^4 \)

- First to notice, that \( N \) doesn't matter! \( \times 6 \)

Just look at anagrams of \( 111LL \) which contain subsequence \( 1L1L \) & then insert \( N \) back.

What is the condition on anagram of \( 111LL \) to contain \( 1L1L \) \( \iff \) \( I + \text{anagram of } \overbrace{111L}^{y!} \)

\[
\frac{y!}{2!2!} = \binom{y}{2} = 6
\]

\[ \Rightarrow 6 \times 6 = 36 > 12 \]
Poker Hands

A hand is a subset of 5 cards.

\[ \binom{52}{5} = \frac{52!}{5!47!} = \frac{52 \times 51 \times 50 \times 49 \times 48}{1 \times 2 \times 3 \times 4 \times 5} \]

Examples:

1) A = \( \frac{1}{2} \) hand contains 703

\[ |A| = \binom{51}{4} \quad P(A) = \frac{|A|}{101} = \frac{\frac{51 \times 50 \times 49 \times 48 \times 47}{1 \times 2 \times 3 \times 4 \times 5}}{101} \]

2) \( P(2 \text{ pairs}) = \frac{|A|}{101} \)

|A|, A = \( \frac{1}{2} \) hands with 2 pairs

\( = \frac{5}{52} \leftrightarrow \text{This is an easier way to compute it!} \)

- Pick 3 different values out of 13
- Choose which of them is single
- Choose suit for each of pairs
- Choose suit for single

\[ \left( \frac{13}{2} \right) \times 3 \times 6 \times 6 \times 4 \times \frac{13 \times 12 \times 11}{1 \times 2 \times 3} \times 3 \times 6 \times 6 \times 4 = 123,552. \]
Last combinatorial enumeration problem.

Rooks attack along vertical & horizontal lines.

\( Q = \{ \) non-attacking rook placements \( \} \)

\( \Leftrightarrow \) permutations of 8 objects:

\[ |Q| = 8! \]

Q: What is the probability that all 8 rooks are at most 1 step away from diagonal?

\( A = \{ \) all rooks are non-att. & within blue region \( \}

= \frac{1}{2} \) permutations of \( \sigma_1, \sigma_2, \ldots, \sigma_8 \):

\[ |\sigma_k - k| < 1 \]

\# of placement of NAR in box of size 1:

\( F_1 = 1 \)

\( F_2 = 2 \)

\( F_3 = 7 \)

\( F_4 = 31 \)

\( F_5 = 122 \)

\( F_6 = 505 \)

\( F_7 = 2187 \)

\( F_8 = 8587 \)

\[ P(A) = \frac{|A|}{2^8} = \frac{34}{81} \]
Infinite Sample Spaces

Reminder: there are different kinds of infinities: countable, the easiest.

Schroeder-Bernstein theorem: infinities can be ordered.

Finite sets:

Countable (infinite) set:

Size of $A$ is no larger than size of $B$ if $f: A \rightarrow B$

there is a injective $f : A \rightarrow A'$ 

$|A| \leq |B|$, $|B| \leq |A| \Rightarrow |A| = |B|$
Cantor's theorem: set of subsets is bigger than the set itself.

Rational or algebraic numbers form a countable set; real numbers are bigger.

Assume there is \( 1-1 \) corr. \( A \) \& \( 2^A \)

\[ f: A \to 2^A : f(a) \subseteq A \]
\[ g: 2^A \to A : (g(S)) = \{ a \mid a \notin f(a) \} \]

Consider subset \( S \subseteq A : S = \{ a : a \notin f(a) \} \)

Take \( a = g(S) \subseteq A \), \( f(a) = f(g(S)) = S \)

If \( a \in S \) then \( a \notin f(a) = S \)

If \( a \notin S = f(a) \) then \( a \in S \)
Cantor's theorem: set of subsets is bigger than the set itself.

Rational or algebraic numbers form a countable set; real numbers are bigger.

\[ \mathbb{N} \times \mathbb{N} \leftrightarrow \mathbb{N} \]

\[
\begin{array}{cccc}
(1,1) & (1,2) & (1,3) & (1,4) \\
2 & 3 & 4 & 5 \\
3 & 4 & 5 & 6 \\
\end{array}
\]

Countable:

\[ k \rightarrow k/2 \text{ if } k \text{ is even} \]

\[ k \rightarrow (k+1)/2 \text{ if } k \text{ is odd} \]

\[ 1 \rightarrow 1' \quad \text{But: Real #s are uncountable.} \]

Rationals are countable!

\[ \sqrt{2} \quad 3\sqrt{5} - \sqrt{3} \leq 10253 \]

\[ \begin{array}{c}
\text{Take binary note: } \\
\end{array}
\]

\[ 0.10110100 \leftrightarrow \text{Subsets of } \mathbb{N} \]

\[ 9.3 \]
Examples of Countable Probability Spaces

\( (Q, \mathcal{F}, P) \)

Without any paradoxes, \( \mathcal{F} = 2^Q \)

\[ P(A) = \sum P(\omega_i) \]

\[ p_k = P(\omega_k) \geq 0 \]

\[ P(Q) = \sum_{k=1}^{\infty} P(\omega_k) = \sum_{k=1}^{\infty} p_k = 1 \]

- Example: \( p_k = P(\omega_k) = 2^{-k} \), \( P(\omega_1) = \frac{1}{2} \), \( P(\omega_2) = \frac{1}{4} \).

\[ \sum 2^{-k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1 \quad \leftarrow \text{Interpretation: # of tosses till you get H.} \]

\[ P(\text{first head happens on even toss}) = \frac{1}{2} \]

\[ p_k \text{ tosses happen: } \overbrace{\text{TTT...TH}} \]

**Puzzle:** How do get event with prob. \( \frac{2^{-k}}{4} \) if you have just a fair coin.
Examples of Countable Probability Spaces

Generalization of bernoulli example: \( P_k = p^{k-1}q \)

Tossing (unfair) coin: prob of success = \( q \), failure = \( p \)

\[
\sum_{k=1}^{\infty} P_k = \sum_{k=1}^{\infty} p^{k-1}q = q \sum_{k=1}^{\infty} p^{k-1} = q \sum_{l=0}^{\infty} p^l = q \cdot \frac{1}{1-p} = 1
\]

Example: \( P_k = \frac{1}{k(k+1)} \)

To check: \( \sum P_k = \left( \frac{1}{2} - \frac{1}{2} \right) + \left( \frac{1}{3} - \frac{1}{3} \right) + \cdots \)

Alternative way to describe how \( \omega \) happens:

\[
\left\{ \begin{array}{ll}
\omega & \text{if } x = \frac{1}{20}, \quad \text{where } y \in [0, \pi] \\
\frac{y_1}{2} & \text{if } x = \frac{1}{3},
\end{array} \right.
\]
Examples of Uncountable Probability Spaces

- Unit Interval

  Lebesgue measure \( \lambda([a, b]) = b - a \).

- Unit Square

  \[ P(A) = \frac{1}{4} + \frac{2}{16} + \frac{4}{64} + \ldots + \frac{2^{k-1}}{4^k} \]

  \[ = \frac{1}{4} \left( 1 + \frac{2}{4} + \frac{4}{16} + \frac{2^l}{4^l} \right) \]

  \[ = \frac{1}{4} \left( 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^l} + \ldots \right) \]

  \[ = \frac{1}{4} \left( \sum_{k=0}^{\infty} \frac{2^k}{4^k} \right) = \frac{1}{4} \left( \sum_{k=0}^{\infty} \frac{1}{2^k} \right) \]

  \[ = \frac{1}{4} \cdot 2 = \frac{1}{2} \]
random variables

A random variable is a real-valued function on $\Omega$.

$x$ — r.v.

$X(w)$ what you observe if $w$ happens.

$\{\omega : X(\omega) \in [a, b]\} = X^{-1}([a, b]) \in \mathcal{F}$