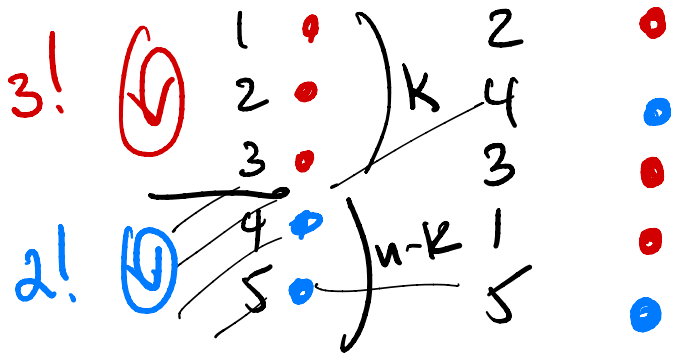
• Enumerated them using symmetries

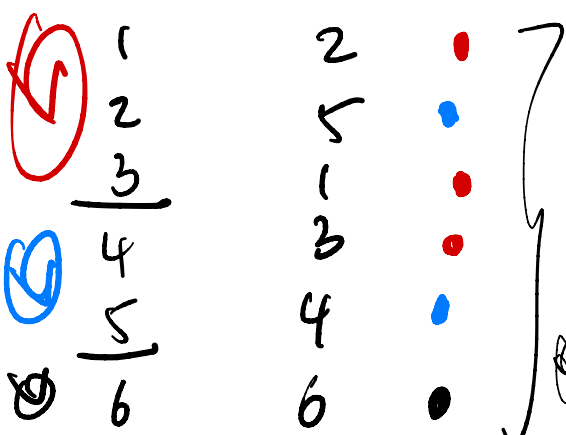
Elementary Enumerative Combinatorics

Examples

Binomial Coefficients



$$= \frac{\text{Total \# all permutations}}{\text{permutations within colors}} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$



Splitting into 3 subsets (of sizes 3, 2, 1)

$$= \frac{6!}{3!\,2!\,1!} = \frac{6\cdot5\cdot4\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{3}\cdot\cancel{2}\cdot\cancel{1}\cdot2\cdot\cancel{1}\cdot\cancel{1}} = 60$$

$$\text{\# of such splits} = \frac{n!}{k!\,l!\,(n-k-l)!} = \binom{n}{k\ l}$$

Symmetries $\longrightarrow$ $\binom{n}{k_1\ k_2\ \cdots\ k_{s-1}} =$

Elementary Enumerative Combinatorics

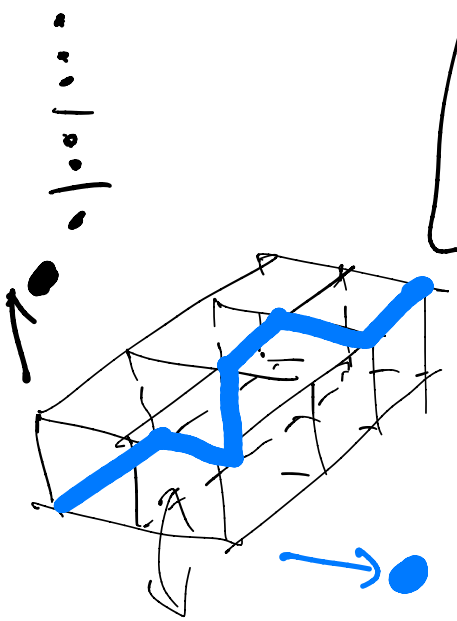
$$= \frac{n!}{k_1!\,k_2!\,\cdots\,k_{s-1}!\,(n-k_1-k_2-\cdots-k_{s-1})!}$$

Examples

Binomial Coefficients

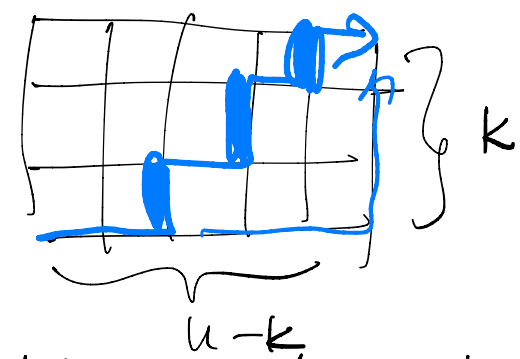
For example, if you want to split 13 elements into 6 groups of sizes 2, 2, 1, 4, 1, 3, the total # of such splits is $\frac{13!}{2!\,2!\,1!\,4!\,1!\,3!} = 1{,}201{,}200$

6



6

n int k & $n-k$



partition into subsets of sizes 3, 2, 1

$(x+y+z)^6 = \ldots$

$\ldots + 60\,x^3 y^2 z^1 + \ldots$

$(x+y+z+u+v)^{15}$

$= \_\_\_\_ \binom{15}{\ldots}$

Anagrams

: I I I L L N

I L L I N I

Shufflings of the letters are called anagrams, & the # of anagrams is given by the multinomial coefficients.

How many anagrams of ILLINI? 60

BBEENOOOORTTT

$B^2E^2NO^4RT^3$

# of anagrams is 1,201,200

One of them is

TO BE OR NOT TO BE?

Examples of computing probabilities in anagram spaces

Anagrams

a) G_{ILLINE} $\supset A = \{$anagrams which have a subword ILL$\}$

$|A| = 12$ $\quad A = A_1 \cup A_2 \cup A_3 \cup A_4$ $\quad A_1 = \{ILL***\}$

$\{*ILL**\}$

$\{**ILL*\}$

$\{***ILL\}$

whose ILL starts

disjoint

$$|A| = \sum_{k=1}^{4} |A_k|$$

ILLIN

These three letters are an anagram of IIN; there are $\frac{3!}{2!\,1!} = 3$ of them

ILL... $\times 3$

.ILL.. $\times 3$

$\times 3$

$\times 3$

$|G_{MINI}| =$

$$= \frac{4!}{2!\cdot 1!\cdot 1!} = 12$$

Anagrams

Another version b) How many anagrams will have a subsequence $\times I \times L \times L \times$

- First to notice, that N doesn't matter! $\times 6$

$\boxed{I\ I\ \underline{L}\ N\ I\ \underline{L}}$

Just look at anagrams of IIILL which contain subsequence I L L & then insert N back

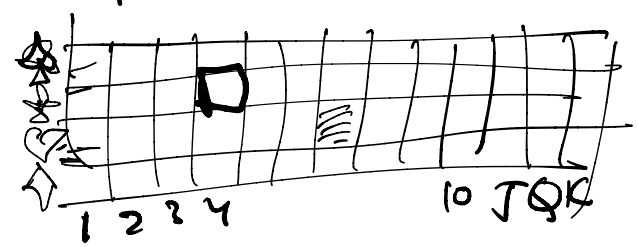
What is the condition on anagram of IIILL to contain $\cdot I \cdot L \cdot L \cdot$ $\Longleftrightarrow$ I + anagram of $\underbrace{IILL}$

$\frac{4!}{2!2!} = \binom{4}{2} = 6$

$\Longrightarrow \quad 6 \times 6 = 36 > 12$

Deck has 52 cards, 13 values, 4 suites

Poker Hands A hand is a subset of 5 cards.



$$|\Omega| = \binom{52}{5} = \frac{52!}{5!\,47!} = \frac{52\cdot 51\cdot 50\cdot 49\cdot 48}{1\cdot 2\cdot 3\cdot 4\cdot 5}$$

<u>Examples</u> a) $A = \{\text{hand contains } 7\heartsuit\}$

$$|A| = \binom{51}{4} \qquad \mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{\cancel{1\cdot 2\cdot 3\cdot 4\cdot 5}}{\cancel{48\cdot 49\cdot 50\cdot 51\cdot 52}} \cdot \frac{\cancel{51\cdot 50\cdot 49\cdot 48}}{\cancel{1\cdot 2\cdot 3\cdot 4}}$$

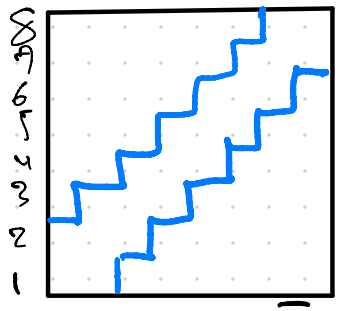
b) $\mathbb{P}(\text{2 pairs}) = \frac{|A|}{|\Omega|}$ $= \frac{5}{52}$ ← There is an easier way to compute it!

$|A|$, $A = \{\text{hands with 2 pairs}\}$

- Pick 3 different values out of 13
  - Choose which of them is single
    - Choose suite for each of pairs
    - Choose suit for single

$$\binom{13}{3} \times 3 \times 6 \times 6 \times 4 = \frac{13\cdot 12\cdot 11}{1\cdot 2\cdot 3} \times 3 \times 6 \times 6 \times 4 = 123{,}552.$$

Last combinatorial enumeration problem.



Rooks attack along vertical & horizontal lines

$\Omega = \{$ non-attacking rook placements $\}$

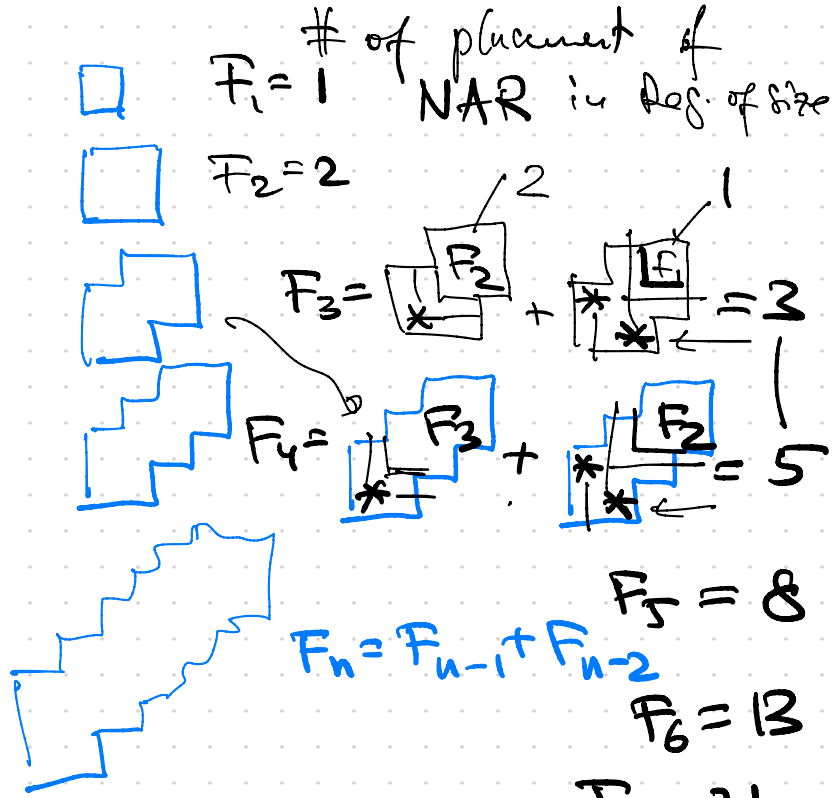
$\Longleftrightarrow$ permutations of 8 objects:

$|\Omega| = 8!$

Q: What is the probability that all of 8 rooks are at most 1 step away from diagonal

$A = \{$ all rooks are non-att. & within blue region $\}$

$= \{$ permutations $\sigma_1 \sigma_2 \dots \sigma_n$: $|\sigma_k - k| \leq 1$



$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{34}{8!}$$

Infinite Sample Spaces

Reminder: there are different kinds of infinities: countable, the easiest.

Schroeder-Bernstein theorem: infinities can be ordered.

Two sets have the same size if there is a 1-1 correspondence between them

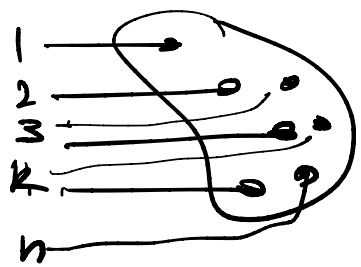
$f: A \to B$: different el'ts go to diff. elements

$A \leftarrow B : g$

$g(f(a)) = a$

$f(g(b)) = b$

Finite sets:



"size"

less or equal than

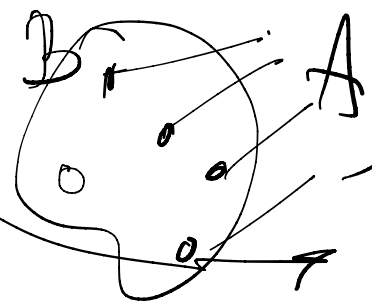
$|A| \leq |B|$ "size"

Countable (infinite) set:

$A \longleftrightarrow \mathbb{N} = \{1, 2, 3, \ldots\}$

$\|$

$\{a_1, a_2, a_3, a_4, \ldots a_k, \ldots\}$



Size of A is no larger than size of B if $f: A \to B$ there is a injective f: $a \neq a' \Rightarrow f(a) \neq f(a')$

<u>Thm</u> $|A| \leq |B|$, $|B| \leq |A| \Rightarrow |A| = |B|$

$|A| = 3 \quad 2^3 = 8$

$\{a, b, c\} \quad 2^A = \{\emptyset, \{a\}, \{b\} \ldots\}$

Cantor's theorem: set of subsets is bigger than the set itself.

$A \leftarrow$ any set

$2^A \leftarrow$ set of subset of A

$|A| \leq |2^A| \quad : f(a) = \{a\}$

Rational or algebraic numbers form a countable set; real numbers are bigger.

Assume there is 1-1 corr. btw A & 2^A

$f : A \to 2^A \quad : f(a) \subset A$

$g : 2^A \to A \quad f(g(S)) = S \; ; \; g(f(a)) = a$

For $A = \mathbb{N}$, if $2^{\mathbb{N}}$ is countable the

	1	2	3	4	5	6	
S_1	0	1	1	0	0	1	
S_2	0	0	1	1	1	0	..
S_3	0	1	1	1	0	1	
	0	1	1	1	1	1	

Consider subset $S \subset A : S = \{a : a \notin f(a)\}$

Take $a = g(S) \in A$. $f(a) = f(g(S)) = S$

If $a \in S$ then $a \notin f(a) = S$

If $a \notin S = f(a)$ then $a \in S$

$\mathbb{N} \cup \mathbb{N} \cup \mathbb{N} \cup \mathbb{N} \longleftrightarrow \mathbb{N}$

Cantor's theorem: set of subsets is bigger than the set itself.

Rational or algebraic numbers form a countable set; real numbers are bigger.

$\mathbb{N} \times \mathbb{N} \longleftrightarrow \mathbb{N}$

$(1,1)\ (1,2)\ (1,3)\ (1,4)$

$(2,1)\ (2,2)\ (2,3)$

$(3,1)\ (3,2)\ (3,3)$

$\Rightarrow |\mathbb{Q}| = |\mathbb{N}|$

Rationals are countable!

$\sqrt{2}$ $\quad \sqrt[3]{5-\sqrt{3}} \leftarrow 10253$

$\left\{ \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1' & 2' & 3' & 4' & 5' & \end{matrix} \right\}$

countable:

$k \mapsto k/2$ if k is even

$k \mapsto ((k+1)/2)'$ if odd

$1 \to 1'$

$2 \to 1$

$3 \to 2'$

$4 \to 2$

But: real #s are uncountable.

Take binary notation for real #s $0 \le r \le 1$

$r = .10110100$

$\Longleftrightarrow$ subsets of $\mathbb{N}$

9.3

$(\Omega, \mathcal{F}, \mathbb{P})$

Examples of Countable Probability Spaces

sample

elementary event

$|\Omega| = |\mathbb{N}| \iff \Omega = \{\omega_1, \omega_2, \omega_3, \dots$

Without any paradoxes, $\mathcal{F} = 2^{\Omega}$

$A = \{\omega_{i_1}, \omega_{i_2}, \dots\}$

$$\mathbb{P}(A) = \sum \mathbb{P}(\omega_{i_k})$$

$$A = \bigcup_{k=1} \{\omega_{i_k}\}$$

$$p_k = \mathbb{P}(\omega_k) \geq 0$$

$$\mathbb{P}(\Omega) = \sum_{k=1}^{\infty} \mathbb{P}(\omega_k) = \sum_{k=1}^{\infty} p_k = 1$$

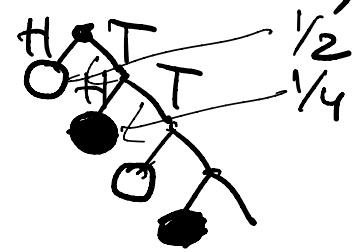
- <u>Example</u> $p_k = \mathbb{P}(\omega_k) = 2^{-k}$ $\mathbb{P}(\omega_1) = 1/2$ $\mathbb{P}(\omega_2) = 1/4$.

$$\sum 2^{-k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1$$

$\leftarrow$ Interpretation: # of tosses till you get H.

$\mathbb{P}$(first head happens on even toss) $= 1/3$

k tosses happen $\underbrace{TTT\dots T}_{k-1}H$



<u>Puzzle</u> How to get event with prob. if you have just a fair coin

Examples of Countable Probability Spaces

$$\boxed{\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \dots = \frac{1}{1-x}, \quad \text{if } |x| < 1.}$$

Generalization of previous example $\quad p_k = p^{k-1} \cdot q \qquad p_k = \mathbb{P}(\omega_k) = \{\text{first success at } k\}$

Tossing (unfair) coin: prob. of success $= q$, failure $= p$

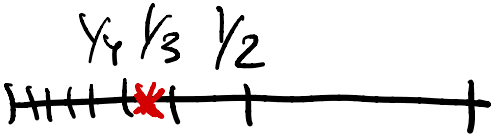
$$\sum_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty} p^{k-1} q = q \sum_{k=1}^{\infty} p^{k-1} = q \sum_{\ell=0}^{\infty} p^{\ell} = q \cdot \frac{1}{1-p} = 1 \qquad p+q=1$$

$\ell = k-1$

<u>Example</u> $\quad p_k = \dfrac{1}{k(k+1)} \qquad p_1 = \frac{1}{2},\ p_2 = \frac{1}{6},\ p_3 = \frac{1}{12}, \dots$

To check $\quad \sum p_k = \underbrace{\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right)}_{p_1} + \underbrace{\left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right)}_{p_2} + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) \dots \qquad p_k = \frac{1}{k} - \frac{1}{k+1}$

$= 1$

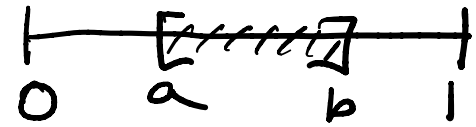


Alternative way to describe how ω_k happens:

if $k = \left\lfloor \frac{1}{\omega} \right\rfloor$, where ω is $0 \le \omega \le 1$

$(\Omega, \mathcal{F}, \mathbb{P})$

Ω



Any interval $[a,b] \in \mathcal{F}$

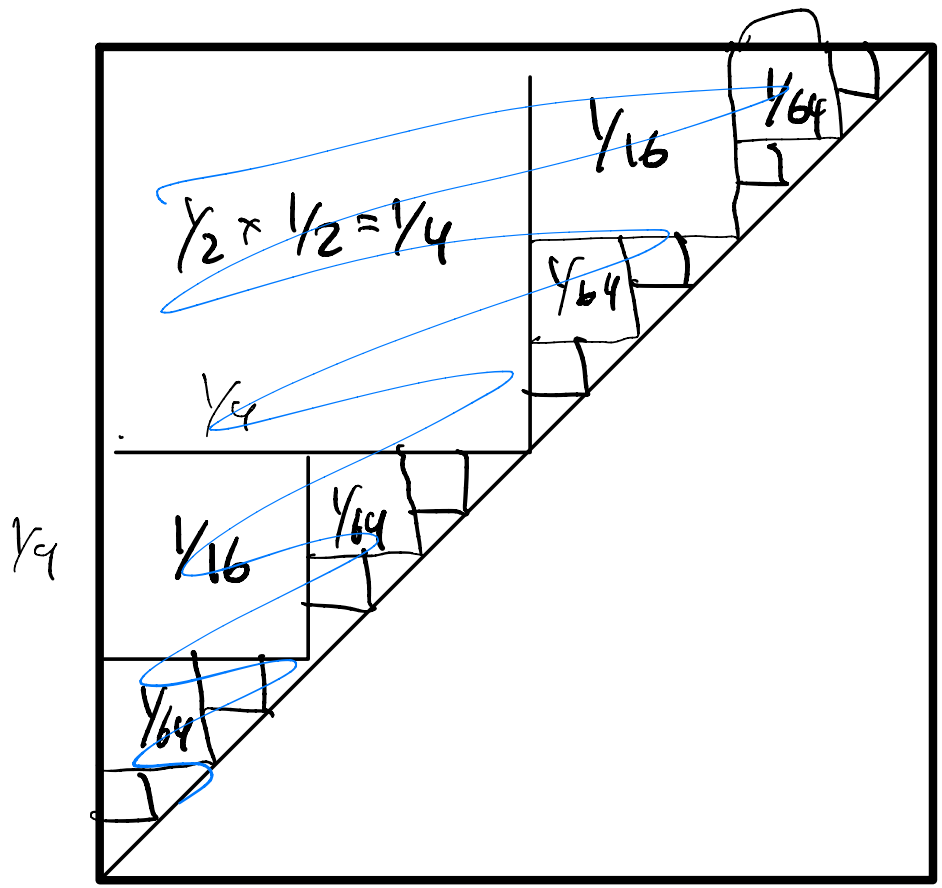
Examples of Uncountable Probability Spaces

- Unit Interval

Lebesgue measure $\mathbb{P}([a,b]) = b - a.$

- Unit Square

$\mathbb{P}(\boxtimes) = (b-a)(d-c)$ $\quad \ell = k-1$



$$\mathbb{P}(A) = \frac{1}{4} + \frac{2}{16} + \frac{4}{64} + \dots \frac{2^{k-1}}{4^k} + \dots$$

$$= \frac{1}{4}\left(1 + \frac{2}{4} + \frac{4}{16} + \dots \frac{2^\ell}{4^\ell} + \dots\right)$$

$$= \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^\ell} + \dots\right)$$

$$= \frac{1}{4}\left(\sum_{k=0}^{\infty} \frac{2^k}{4^k}\right) = \frac{1}{4}\left(\sum \frac{1}{2^k}\right) =$$

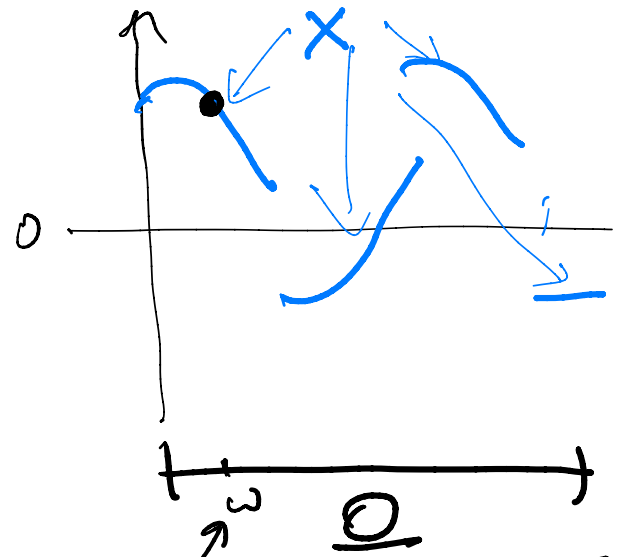
$x = \frac{1}{2}$

$$= \frac{1}{4} \cdot 2 = \frac{1}{2}$$

$\frac{1}{\ldots} = 2$

random variables

A random variable is a real-valued function on Ω



X — r.v. $\quad X(\omega)$ what you observe if ω happens.

[Constraints on r.v. $\{\omega : X(\omega) \in [a,b]\} = X^{-1}([a,b]) \in \mathcal{F}$]

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