idea of probability

We all have an intuitive notion of what the probability means. Intuition, however, often fails us.

Let’s check it: I toss three coins (fair ones!), and declare that at least one is heads. What are the chances now that all three are heads?
Basic notion of probability

is that of sample space, points of which are elementary outcomes. The sample space is usually referred to as \( \Omega \). Subsets of \( \Omega \) are called events. Empty space is an event (impossible). \( \Omega \) is an event (certain).

Event: 3rd digit is 7

\( \Omega \)

Elementary outcomes

\( \$100 \) bill

Elementary event

\( \omega \in \Omega \)

Sample space

1st coin is heads up
Set theory

Notions to internalize:

- union, intersection
- complement
- partition

Algebra of sets.

De Morgan laws.

\[ A^c = \{ \omega : \omega \notin A \} \]
\[ (A^c)^c = A \]

Obligatory (for some reasons) notion: Karnaugh maps

\[ (A \cup B)^c = A^c \cap B^c \]
\[ (A \cap B)^c = A^c \cup B^c \]

A \subseteq B

\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]
\[ A(B \cup C) = AB \cup AC \]
Set theory

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Probability Spaces

Events are not all of the possible subsets of the sample space.
(Wait, why?)

Events are assigned a nonnegative number, the probability.

The data describing the model are called probability space, \((\Omega, \mathcal{F}, P)\).

Set Axioms

- Axiom 1 \(\Omega\) is an event
- Axiom 2 Complement to an event is an event
- Axiom 3 Events form a \(\sigma\)-algebra

Corollaries

\(\mathcal{F}\) is the collection of subsets about which we can reason.

\[ A \in \mathcal{F} \implies A^c \in \mathcal{F} \quad P: A \rightarrow \mathbb{R} \geq 0 \]

\(\sigma\)-Algebra: If you have a sequence of events \(A_1, A_2, \ldots, A_k, \ldots\), then

\[ A_1 \cup A_2 \cup \cdots \cup A_k \in \mathcal{F} \]
Probability Spaces

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Corollaries

Examples of \(\sigma\)-algebras:
- Trivial \(\mathcal{F} = \{\emptyset, \Omega\}\)
  \[\mathcal{F} = \{\emptyset, \Omega, \{0, 1\}, \{0, 3, 4, 5\}\}\]
- Non-trivial \(\sigma\)-algebras
  Take
Probability Axioms

- Probabilities are nonnegative;
- If the events are mutually exclusive, then the probabilities are additive;
- Probability of the sample set $\Omega$ is 1.

Corollaries

- $P(\emptyset) = 0$ (note, in general, $P(A) = 0 \Rightarrow A = \emptyset$)
- If $A_1, A_2, A_3, \ldots$ are mutually exclusive, then
  
  $P(\bigcup_k A_k) = \sum_k P(A_k) = P(A_1) + P(A_2) + \ldots$

- $P(A) + P(A^c) = P(\Omega) = 1 \Rightarrow 0 \leq P(A) \leq 1$

- The sum $\sum P(A_k)$ (for mutually exclusive events) is convergent.
Probability Axioms

- Probabilities are nonnegative;
- If the events are mutually exclusive, then the probabilities are additive;
- Probability of the sample set $\Omega$ is 1.

Corollaries

If \[ \sum_{k} P(A_k) = \sum_{l} P(B_l) \]
then \[ \sum P(A_k) = \sum P(B_l) \]

$P(A_1) + P(A_2) = P(B_1) + P(B_2)$
Probability Axioms

- Probabilities are nonnegative;
- If the events are mutually exclusive, then the probabilities are additive.

Probability of the sample set \( \Omega \) is 1.

\[
\sum P(A_i) = 1
\]

Corollaries

Inclusion – exclusion formula

\[
P(A_1) + P(A_2) = P(A_1 \cup A_2) + \underbrace{P(A_1 \cap A_2)}_{= 0}
\]

\[
P(A_1) + P(A_2) - P(A_1 \cap A_2) = P(A_1 \cup A_2)
\]
Combinatorial probability spaces

Frequent situation when

- sample space is finite;
- all subsets of the sample space are events; and
- the probability is symmetric: probability of any elementary event is the same, $|\Omega|^{-1}$.

Examples

- $P(A) = \frac{|A|}{|\Omega|}$

Counting is the central question in combinatorics.
Elementary Enumerative Combinatorics

Tools:

- Partitioning
- Products
- Symmetries
- Paths Counting
- Combinatorial Isomorphisms

$A$ (as a set) is Cartesian product of $A_1 \times A_2$:

$A = \{(a_1, a_2)\}$
Examples

Partitioning and Negations

Example: \( O = \frac{1}{2} \) words of length 6 in letters \( \alpha, \beta, \gamma, \delta, \varepsilon \in \Omega \)

\[ \text{Bates} \quad c^- \quad \text{Olaf} \quad 56 \quad a^- \quad C. \]

\[ 101 = 56 \]

\[ C = 2 \text{ words with at least } 1 \alpha \text{ and at least } 1 \beta \]

\[ \varepsilon \varepsilon \in \delta \delta \notin C \]

\[ A = 4 \text{ words w } \alpha \beta \gamma \delta \]

\[ B = 3 \quad P^3 \]

\[ |A \cap B| = |(A \cup B)^c|^c | = \]

\[ |(A^c \cup B^c)^c|^c | = \text{O} | - |A^c \cup B^c| = |O| - (A^c) - |B^c| + |A^c B^c| \leq 8.27 \]
Products and Permutations

Set of permutations of $1, 2, 3, \ldots, n$; Permutation is a string of these objects so that each appears once.

$S_n = S_3 = \{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}\}$

$|S_n| = n!$

**Construction:** take elements 1 by 1 & write down their ranks: $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$

$\Gamma_1 = 1$

$\Gamma_2 = 1$

$\Gamma_3 = 2$  

$\Gamma_3 = 3$  

$\Gamma_k = \{1, 2, \ldots, k\}$

Did we miss any permutation?

$P(\text{contestant has a record}) = \frac{1}{4}$
### Binomial Coefficients

\[
\binom{n}{k} = \text{the \# of ways to pick } k \text{ elements out of } n \text{ distinct ones}
\]

\[
\begin{align*}
\text{ABCD} & \rightarrow \text{pick 2 distinct ones} \\
\text{BDAC, AD} & \\
\text{AC} & \\
\text{BDAC} & \\
\text{AD} & \\
\text{DBCA} & \\
\text{BDAC} & \\
\text{DBCA} & \\
\end{align*}
\]

\[
\binom{4}{2} = 6
\]

\[
\text{Algorithm: take a permutation and pick first } k \text{ in it.}
\]

\[
\#	ext{ permutations} = n!
\]

\[
\begin{align*}
4! & = 24 \\
\text{BDAC, DB} & \\
\text{BDAC} & \\
\text{DB} & \\
\text{CA} & \\
\end{align*}
\]

\[
\text{In general all permutations}
\]

\[
\binom{n}{k} = \frac{n!}{k! \ (n-k)!}
\]

\[
\frac{24}{2 \cdot 2} = 6
\]
Newton's Interpretation

Binomial Coefficients

\[(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k + \cdots\]

\[(x+y)(x+y) = (x+y)^2\]

\[\sum_{w} \binom{n-w}{w} = \binom{n}{w}\]

Paths counting

\((4,4)\)

\[\begin{array}{cccccc}
1 & 5 & 15 & 35 & 70 \\
1 & 4 & 10 & 20 & 35 \\
1 & 3 & 6 & 10 & 15 \\
1 & 2 & 3 & 4 & 5 \\
1 & 1 & 1 & 1 & 1 \\
\end{array}\]

\[1 + 6 + 3 = 10\]
Path counting

Binomial Coefficients

\[ \binom{3}{2} = 3 \]

\[ \binom{4}{3} = 4 \]

\[ \binom{5}{4} = 1 \]

\[ \binom{6}{5} = 1 \]

\[ \binom{7}{6} = 1 \]

\[ \binom{8}{7} = 1 \]

\[ \binom{9}{8} = 1 \]

\[ \binom{10}{9} = 1 \]

\[ \binom{11}{10} = 1 \]

such

\# of trajectories that
at any step those more it
than I so far

Catalan numbers

\[ 1, 1, 2, 5, 14, 42, \ldots \]
Anagrams

\[
\begin{array}{c}
\text{ILLINI} \\
\text{IIIILLN} \\
\text{123456}
\end{array}
\]

\[
\frac{6!}{3!2!1!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 1} = 60
\]

Example of comb. Q:

How many of these anagrams have a subway?

NIL  N1111

\[\text{820}\]

The foundations of these operations is evident enough, in fact; but be explanation of it now, I have preferred to conceal it thus: 6accdae13 foundation I have also tried to simplify the theories which concern t arrived at certain general Theorems.