Linear Obstacles in Linear Systems, and How to Avoid Them ⋆

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Abstract: We derive the topological structure of the space of trajectories avoiding a collection of (instantaneous) obstacles that are affine linear subspaces.

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1. MOTIVATION

The problems of motion planning or feedback control in the presence of obstacles - finding the trajectory of an engineered system avoiding the forbidden regions of the configuration space, - are very common in robotics, autonomous systems, and many other areas, too numerous to list here.

1.1 Spaces of Paths

A common theme among the variety of approaches to finding such trajectories is to start with an arbitrary path in the feasible region, and then to minimize some figure of merit (e.g. a cost function depending on some norm of control, on the proximity to the forbidden region as measured by a barrier function, complexity of the control or some combination of those, - see, e.g. Liu et al. (2019); Gauthier and Kawski (2014)).

In other words, one deals with a proper functional on the space of paths avoiding the obstacles (presumably exploding as the trajectory runs close to the forbidden regions), and looks for a minimizer of that functional. As usual one needs to be concerned with the number of critical points of various indices this functional might have.

Morse theory tells us that the critical points of any proper function on an open subset of a manifold are intimately related to the topology of that subset. Hence, it is natural to pose the general question: given a control system on a manifold, perhaps with boundary (to account for forbidden regions), what is the structure of the trajectories of the control system, avoiding these boundaries?

In general, Morse theory combined with some form of the h-principle (Eliashberg et al. (2002)) gives sufficient tools to address this problem. The situation becomes less familiar, when the obstacles are *changing in time*: how one would plan motion involving crossing a traffic light?

1.2 Spaces of Directed Paths

In computer science, this kind of problems occurs when one attempts to handle systematically *concurrency problem*, dealing with forbidden regions in the Cartesian products of execution spaces of several algorithms, which should not be allowed to claim the same resource at the same time. The notion that the topology of the space of *directed paths* is highly relevant to the concurrency problem was realized early and addressed in abundant literature (see, e.g. Fajstrup et al. (2016)).

However, there are few explicit results about the topological structure of the space of directed paths, and they are essentially non-existent in non-discrete settings (for a rare example, see Meshulam and Raussen (2017)).

1.3 Topology of the Spaces of Directed Paths

The directed paths emerging in the context of concurrency theory can be interpreted as the trajectories in $\mathbb{R}^d$ all of whose coordinates are increasing, or, if one assumes differentiability, as the trajectories whose velocities lie in the positive coordinate orthant. Replacing this orthant with a general (open convex) cone already leads to interesting generalizations. In this note we add an extra twist to the problem, considering control systems.

Our aim in this note is introduce in control-theoretic setting the general problem of describing the *topology* of the space of obstacle avoiding paths. (Here by topology we mean the *homotopy type*, the invariant that would in particular bound from below the number of critical points of any smooth functional.)

We will address in this note the special case of linear control systems, with *instantaneous affine-linear obstacles*[1].

As usual, understanding this homotopy type is difficult in general, so that we focus on the characterization of common homotopy invariants, the (co)homology groups of the space.

We will be using throughout the singular (co)homologies with integer coefficients.

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[1] One can find many situations where this modeling assumption is relevant: think of an asteroid whose speed makes the duration of its presence in the flight path region negligible, but whose trajectory should be avoided at all costs...

2. SETUP

Consider a linear control system,

$$\dot{\mathbf{x}} = A\mathbf{x} + B\boldsymbol{u}, \mathbf{x}(0) = \mathbf{x}_* \tag{1}$$

where $\mathbf{x} \in V = \mathbb{R}^d, \boldsymbol{u} \in P$, P a convex open subset of $U = \mathbb{R}^m$. We assume here (without the loss of generality) that the system (1) is controllable.

Further, we consider a sequence of *instantaneous obstacles* $\mathbf{O}_k \subset V, k = 1, \ldots, K$ realized at times $0 < t_1 < t_2 < \ldots < t_k < \ldots < t_K < T$. We will denote these data as $\mathcal{O} = \{(\mathbf{O}_k, t_k)\}_k$.

We will be interested in the trajectories of the system *avoiding the obstacles*:

$$\mathbf{D}_{\mathcal{O}} = \{(\mathbf{x}(\cdot) : \mathbf{x}(t_k) \notin \mathbf{O}_k \text{ for } k = 1, \ldots, K.$$

One can see easily that if the obstacles are closed subsets of V, then the collection of obstacle avoidance trajectories (identified with, say, the collection of the initial conditions and controls $\{\mathbf{x}_*, \boldsymbol{u}(\cdot)\} \subset V \times L_1([0,T], P)$ realizing obstacle avoiding paths) forms an open subset in the space of trajectories.

Throughout this note we will restrict ourselves to the simple situation of *affine-linear* obstacles $\mathbf{O}_k \subset V$.

We will be interested in the *homotopy invariants* of the obstacle-avoiding trajectories of (1).

3. RESULTS

We start with a warm-up model, to illustrate the key notions.

3.1 Point Obstacles on Plane

Consider the simplest possible example:

$$\dot{x} = u, x, u \in \mathbb{R}^1, x(0) = x_*, |u| < 1.$$

Consider a finite collection of 0-dimensional obstacles, - just a sequence of K points on the (t, x) plane ordered left-to-right:

$$\mathcal{O} = \{(t_1, x_1), (t_2, x_2), \ldots, (t_K, x_K)\}, 0 < t_1 < \ldots < t_K.$$

We can interpret the trajectories of this control system as Lipschitz ($L < 1$) functions, and the obstacle-avoiding trajectories are just the functions which don't pass through particular values at the specified times.

It is clear that the obstacle avoiding functions form a finite collection form a finite collection of open convex subsets, so that the question on the topology of $\mathbf{D}_{\mathcal{O}}$ reduces to the question about the number of such components.

Example Consider the collection of obstacles shown on Fig. 1 (here $x_* = 0$). One can easily see that there are 11 connected components of the obstacle-avoiding 1-Lipschitz functions satisfying $x(0) = 0$.

We augment the collection of obstacles with the point $(0, x_*)$, define a *chain* to be a sequence of obstacles (starting with $\mathbf{O}_0 = (0, x_*)$) such that any two of them can be connected by a straight segment in the plane with the slope between -1 and 1 (i.e. $|x_l - x_k| < t_l - t_k$ for $t_k < t_l$). (Of course, these chains are the same as the chains in the partial order induced on the obstacles by $k \prec l \Leftrightarrow |x_l - x_k| < t_l - t_k$.)

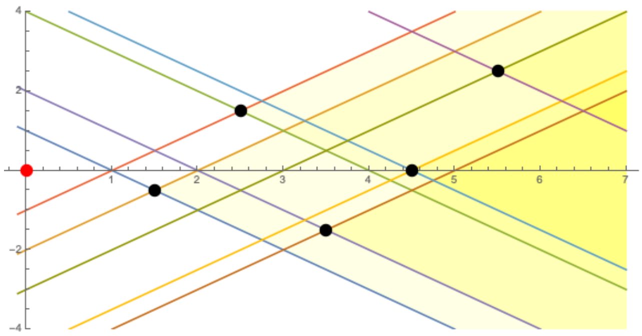

Fig. 1. The simple linear control problem with point-like obstacles. The slope ± 1 functions through the obstacles are shown for visual help.

Theorem 1. The space of obstacle avoiding trajectories in this system is a disjoint union of finitely many contractible components, which are in one-to-one correspondence with the *chains* of the ordering described above starting with $(0, x_*)$.

Proof of Theorem 1 Consider the decomposition of $\mathbf{D}_{\mathcal{O}} = \cup_\gamma \mathbf{D}_{\mathcal{O}}(\gamma)$ into open convex components. For such component, the pointwise infimum of all trajectories in it (recall that we identify the trajectories with Lipschitz functions $x : [0, T] \to \mathbb{R}$) is a Lipschitz function $x_\gamma(t) = \inf_{x \in \mathbf{D}_{\mathcal{O}}(\gamma)} x(t)$ passing through some collection of obstacles (including, necessarily, $\mathbf{O}_0 = (0, x_*)$). Those of the obstacles where x_γ is not locally linear necessarily form a chain, which we will refer to as the *marker* of the component. This gives a mapping from the components of $\mathbf{D}_{\mathcal{O}}$ to the chains. To reverse the correspondence, i.e. to associate to a chain a component, pick a small slack ϵ, and for a chain γ consider the function

$$x_\gamma(t) := \max_{k \in \gamma} (x_k + \epsilon - (1 - \epsilon)|t - t_k|).$$

One can easily see that for small enough ϵ, this function will be in the component of $\mathbf{D}_{\mathcal{O}}$ whose marker coincides with the chosen chain. This bijection proves the statement. $\square$

Returning to the example of Fig. 1: the partial order contains one 1-chain, $\mathbf{O}_0$, five 2-chains, $\mathbf{O}_0 \prec \mathbf{O}_k, k = 1, \ldots, 5$ and five 3-chains $\mathbf{O}_0 \prec \mathbf{O}_1 \prec \mathbf{O}_3, \mathbf{O}_0 \prec \mathbf{O}_1 \prec \mathbf{O}_4, \mathbf{O}_0 \prec \mathbf{O}_1 \prec \mathbf{O}_5, \mathbf{O}_0 \prec \mathbf{O}_2 \prec \mathbf{O}_4, \mathbf{O}_0 \prec \mathbf{O}_2 \prec \mathbf{O}_5$ and no longer chains.

3.2 Main Result

Our main result deals with the situation where all obstacles are *affine linear* subspaces of V.

We need some preliminary definitions.

Avoidance classes The space V from which an l-dimensional affine linear subspace $\mathbf{O}$ is removed is homotopy equivalent to the $c = d - l - 1$-dimensional sphere S^c. In particular, its c-dimensional cohomology group is cyclic, and generated by the class Poincare-dual to any properly oriented $(l + 1)$-dimensional affine half-plane having $\mathbf{O}$ as its boundary. We will be referring to this class as the *avoidance class* $a \in H^c(V - \mathbf{O})$.

Evaluating a trajectory in $\mathbf{D}_{\mathcal{O}}$ at t_k generates a mapping

$$e_k : \mathbf{D}_{\mathcal{O}} \to V - \mathbf{O}_k,$$

which pulls the avoidance classes to the space $\mathbf{D}_{\mathcal{O}}$, generating the classes

$$\alpha_k = H(e_k)^*(a_k) \in H^{c_k}(\mathbf{D}_{\mathcal{O}})$$

(here $c_k = d - \dim \mathbf{O}_k$ is the codimension of k-th obstacle).

Chains of obstacles Generalizing the example of Section 3.1, we introduce *chains of obstacles.* Namely, an (ordered) sequence of indices $0 < k_1 < k_2 < \ldots < k_L$ forms a *chain* if there exists a trajectory of (1) starting at at x_* and passing through all of the obstacles in the chain, for some admissible control $u(\cdot)$ in P.

Viewed as a system of subsets of the collection of obstacles, the chains form, clearly, a simplicial complex (subset of a chain is a chain), which we will denote as $\mathbf{C}_{\mathcal{O}}$.

Example Consider the simple *double integrator* system,

$$\dot{x} = v, \dot{v} = u; |u| \leq A \tag{2}$$

and a collection of the codimension 2 obstacles $\mathbf{O}_k = (x_k, v_k; t_k)$. Then the obstacles form a chain if for any pair of adjacent indices $k < k'$ in the sequence $0 < k_1 < k_2 < \ldots$, forms a chain, which leads to the natural compatibility condition: $\mathbf{O}_k, \mathbf{O}_{k'}$ is a chain *iff* the region in (t, x) plane given by

$$\begin{aligned} t_k < t < t_{k'} \\ x_k + v_k(t - t_k) - A(t - t_k)^2/2 < x; \\ x_k + v_k(t - t_k) + A(t - t_k)^2/2 > x; \\ x_{k'} + v_{k'}(t - t_{k'}) - A(t - t_{k'})^2/2 < x; \\ x_{k'} + v_{k'}(t - t_{k'}) + A(t - t_{k'})^2/2 > x \end{aligned}$$

is path-connected. In particular, the condition on the obstacles to form a chain is semi-algebraic.

Figure 2 presents an example of 3 obstacles (only (t, x) coordinates shown). The obstacles $\mathbf{O}_1$ at $t_1 = 0$, $\mathbf{O}_2$ at $t_2 = 3$, $\mathbf{O}_3$ at $t_3 = 4$ form two chains of length 2, $\mathbf{O}_1 \prec \mathbf{O}_2$ and $\mathbf{O}_1 \prec \mathbf{O}_3$. As the obstacles have dimension 0, a sequence of obstacles forms a chain iff any two neighboring obstacles form a chain. This is not true for the obstacles of higher dimensions.

In general, the conditions on a sequence of obstacles to form a chain are not semialgebraic and can be quite complicated.

Cohomology of the Space of Obstacle Avoiding Trajectories Our main result describes completely the cohomology ring structure of the obstacle aviding trajectoris of (1) : it is generated by the avoidance classes, and is subject by simple relations:

Theorem 2. The cohomology ring $H^*(\mathbf{D}_{\mathcal{O}})$ is generated by the classes $\alpha_k = H(e_k)^*(a_k)$ subject to relations

$$\alpha_{k_1} \smile \ldots \smile \alpha_{k_L} = 0$$

unless the sequence $k_1 < \ldots < k_L$ forms a chain.

Remark In other words, the cohomology ring $H^*(\mathbf{D}_{\mathcal{O}})$ is isomorphic to the Stanley-Reisner ring of the simplicial complex $\Delta_{\mathcal{O}}$ factored by the squares of the generators.

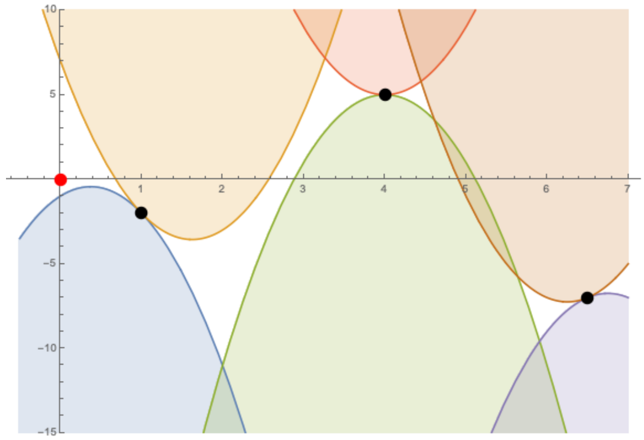

Fig. 2. Two obstacles form a chain if the region bounded by the parabolas with given contact elements at the obstacles is connected.

In the situation of the example of Section 3.1, all generators are in dimension 0, so that the nontrivial elements of the cohomology ring span the space of functions constant on the components of $\mathbf{D}_{\mathcal{O}}$.

4. OUTLINE OF THE PROOF

In this section we sketch the proof of our main result, Theorem 2.

4.1 Finite-dimensional approximation

While the space $\mathbf{D}_{\mathcal{O}}$ is an infinite-dimensional (Banach) manifold, one can easily derive its finite-dimensional approximation. We consider the (augmented) set of instances of obstacles as a mesh $\mathcal{T} := \{0 < t_1 < \ldots < t_K$ subdividing the domain $[0, T]$.

Consider the evaluation map taking the space $\mathcal{S}$ of solutions to (1) to $V^{\mathcal{T}}$: it sends a trajectory $\mathbf{x}(\cdot) \in \mathcal{S}$ to the collection of its values at times $t_k \in \mathcal{T}$. We will denote its image as $\mathcal{B}_{\mathcal{T}}$.

Proposition 3. The evaluation map is a fibration with contractible fibers. In particular, its image $\mathcal{B}_{\mathcal{T}}$ is a contractible open ball in $V^{\mathcal{T}}$, and its restriction to $\mathbf{D}_{\mathcal{O}}$ a homotopy equivalence to its image.

4.2 Alexander duality

For each obstacle $\mathbf{O}_k$, we consider the (convex) set of solutions of (1) that *pass* through that obstacle. This is the intersection of the space of solutions to (1) with the preimage of $\mathbf{O}_k$ under the evaluation map e_k, a closed affine linear subspace of the space of trajectories. We will denote this subspace as $\mathbf{D}(\mathbf{O}_k)$, and its image in $\mathcal{B}_{\mathcal{T}}$ as $\mathcal{B}_{\mathcal{T}}(\mathbf{O}_k)$.

Now, by the Proposition 3, the space $\mathbf{D}_{\mathcal{O}}$ is homotopy equivalent to the complement in the open convex disk $\mathcal{B}_{\mathcal{T}}$ to the union of the (affine-linear) subspaces:

$$\mathbf{D}_{\mathcal{O}} \cong \mathcal{B}_{\mathcal{T}} - \mathcal{B}_{\mathcal{T}}(\mathbf{O})$$

where $\mathcal{B}_{\mathcal{T}}(\mathbf{O}) := \cup_k \mathcal{B}_{\mathcal{T}}(\mathbf{O}_k)$.

In this situation one can deploy Alexander duality. We will identify the ambient space $\mathbb{R}^D \supset \mathcal{B}_{\mathcal{T}}$ of the finite-dimensional approximation of the space of trajectories with the punctured D-sphere. We denote by $K := S^D - \mathcal{B}_\mathcal{T}$ the compact complement to the open disk $\mathcal{B}_\mathcal{T}$. Then

$$\mathcal{B}_\mathcal{T} - \mathcal{B}_\mathcal{T}(\mathbf{O})) \cong S^D - (K \cup \mathcal{B}_\mathcal{T}(\mathbf{O})),$$

whence

$$\tilde{H}_*(\mathbf{D}_\mathcal{O}) \cong \tilde{H}^{D-*-1}(K \cup \mathcal{B}_\mathcal{T}(\mathbf{O})),$$

(here $\tilde{H}$ stands for reduced (co)homologies).

Further, we will notice that

$$\tilde{H}^{D-*-1}(K \cup \mathcal{B}_\mathcal{T}(\mathbf{O})) \cong H^{D-*-1}(K \cup \mathcal{B}_\mathcal{T}(\mathbf{O}), K), \quad (3)$$

as K is contractible.

4.3 Arrangements of subspaces and homotopy colimits

Now the problem reduces to the setup of the homotopy colimits of the arrangements of spaces (see Ziegler and Živaljević (1993)).

The pair of spaces $(K \cup \mathcal{B}_\mathcal{T}(\mathbf{O}), K)$, by the Projection lemma (all the results are from Ziegler and Živaljević (1993)) is homotopy equivalent to the homotopy colimit of the arrangement of spaces

$$\{(K \cup \mathcal{B}_\mathcal{T}(\mathbf{O}_{\boldsymbol{c}}), K)\}_{\boldsymbol{c} \in \mathbf{C}},$$

where for a chain of obstacles $\boldsymbol{c}$

$$\mathcal{B}_\mathcal{T}(\mathbf{O}_{\boldsymbol{c}}) = \cap_{\mathbf{O} \in \boldsymbol{c}} \mathcal{B}_\mathcal{T}(\mathbf{O}),$$

and the mappings

$$d_{\boldsymbol{c}',\boldsymbol{c}} : (K \cup \mathcal{B}_\mathcal{T}(\mathbf{O}_{\boldsymbol{c}'}), K) \to (K \cup \mathcal{B}_\mathcal{T}(\mathbf{O}_{\boldsymbol{c}}), K)$$

are the natural embeddings defined whenever $\boldsymbol{c} \subset \boldsymbol{c}'$.

(Here the diagram of spaces coincides with the collection of chains of obstacles, ordered by inclusion.)

As all the spaces $(K \cup \mathcal{B}_\mathcal{T}(\mathbf{O}_{\boldsymbol{c}'}), K)$ are homotopy equivalent to spheres, and the embeddings $d_{\boldsymbol{c}',\boldsymbol{c}}$ are homotopy equivalent to the constant maps, we arrive, via the Wedge lemma, at the homotopy equivalence

$$(K \cup \mathcal{B}_\mathcal{T}(\mathbf{O}_{\boldsymbol{c}}), K) \cong (\bigvee_{\boldsymbol{c} \in \mathbf{C}} \|\mathbf{C}_{<\boldsymbol{c}}\| * S^{d(\boldsymbol{c})}, *), \quad (4)$$

where $d(\boldsymbol{c}) = \sum_{s \in \mathcal{T}} d_s(\boldsymbol{c})$ is the dimension of the subspace of the (discretized) trajectories passing through the obstacles in $\boldsymbol{c}$: here $d_s(\boldsymbol{c}) = \dim(\mathbf{O}_k) =: d_k$ if $k \in \boldsymbol{c}$, and d otherwise, and $\|\mathbf{C}_{<\boldsymbol{c}}\|$ is the geometric realization of the subcomplex of $\mathbf{C}$ spanned by the chains strictly contained in $\boldsymbol{c}$.

The instantaneous character of the obstacles implies the simplification in (4):

Proposition 4. The affine subspaces $\mathcal{B}_\mathcal{T}(\mathbf{O}_k)$ in $\mathcal{B}$ are either disjoint, or intersection transversally.

In particular, each of the spaces $\|\mathbf{C}_{<\boldsymbol{c}}\|$ is homeomorphic to $S^{|\boldsymbol{c}|-2}$: here $|\boldsymbol{c}|$ is the length of the chain $\boldsymbol{c}$. (By convention, $S^{-1} = \emptyset$.)

Collecting all elements together, we arrive at the

Proposition 5. One has homotopy equivalence

$$(K \cup \mathcal{B}_\mathcal{T}(\mathbf{O}), K) \cong \bigvee_{\boldsymbol{c} \in \mathbf{C}} S^{D - \sum_{k \in \boldsymbol{c}} (c_k - 1) - 1},$$

(where $c_d = d - d_k$ is the codimension of $\mathbf{O}_k$), and therefore the reduced cohomologies of $\mathbf{D}_\mathcal{O}$ is spanned by the classes corresponding to

Finishing the Proof of the Main Result It is easy to see that for each chain of obstacles there exist an embedded a cycle (obtained by chaining small spheres around the obstacles) that evaluates to one exactly on the product of the avoidance classes corresponding to the chain, and vanishes on all other products. This implies that the free abelian group generated by the products of the avoidance classes is embedded into the cohomology ring of $\mathbf{D}_\mathcal{O}$. Combining this with the Proposition 5 implies the claim.

5. EXAMPLES AND CONCLUSIONS

5.1 Examples

Codimension 1 obstacles In this case the spaces is a union of convex open components, and the only question is to enumerate those components. Our main result identifies the free abelian group generated by these components with the chains of obstacles, - the observation which can be observed by the elementary means in the warm-up example in the Section 3.1.

However, the result remains true in more complicated situations as well. Consider, for example, the double integrator example, and replace the obstacles with the condition of simply passing through the points (removing the slope condition). This will turn them into the codimension 1 obstacles. On one hand, this simplifies the topology of the components; on the other hand, the condition that a collection of obstacles forms a chain becomes more intricate, and involves interactions of obstacles beyond neighboring ones. Namely, a collection of obstacles forms a chain if there exists a quadratic spline (with the bound A on the absolute value of the second derivative) passing through the points of the chain. In the example shown on Fig. 2, such a spline exists for any collection of points, and therefore the number of the connected components is 8.

Double Integrator, Codimension 2 In the example (2), the obstacles are of codimension 2 (to hit it, the trajectory has to pass through the point, and match the slope), and so all the primitive avoidance classes $\mathbf{O}_k$ are of dimension 1. One can see that there are two nontrivial products of those classes (as evidenced by the fact there are 2 chains of length 3), so that the Poincare polynomial is

$$\sum_k \mathbf{rk} H^m(\mathbf{D}_\mathcal{O}) t^m = 1 + 3t + 2t^2.$$

In fact, one can see that $\mathbf{D}_\mathcal{O}$ is homotopy equivalent to the Cartesian product of the wedge of two circles and S^1.

5.2 Concluding Remarks

The main theorem remains true (and its proof remains essentially the same) for *time-dependent* linear systems $\dot{\mathbf{x}} = A(t)\mathbf{x} + B(t)\boldsymbol{u}$, as long as they remain controllable over the any time subinterval containing more than one obstacle time.

Similarly, our theorem can be generalized to a significantly broader class of the obstacles (including the non-instantenous ones). Necessary modification would take much more room however, so we postpone them, as well as the detailed proofs of the results presented here to a later publication.

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