Exchangability & Reinforcement

Polya urn: Markov process, with state \((k, l)\), \(k, l \geq 1\), integer.

Transition: \((k, l) \rightarrow (k+1, l)\). Interpretation: urn will

\[
\begin{align*}
\frac{k}{k+l} & \quad \text{pick one at random, return} \\
\frac{l}{k+l} & \quad \text{2 of the same color}
\end{align*}
\]

Proposition: \(X_n = \frac{k}{k+l}\) is a martingale

Indeed,

\[
\mathbb{E}(X_{n+1} | X_n) = \frac{k_n}{k_n + l_n} \cdot \frac{k_n + 1}{k_n + l_n + 1} + \frac{l_n}{k_n + l_n} \cdot \frac{k_n}{k_n + l_n + 1} = \frac{k_n(k_n + l_n + 1)}{(k_n + l_n)(k_n + l_n + 1)} = X_n.
\]

Thus: Bounded martingales converge a.s. [Won't prove]
Hence, $X_n \rightarrow X$, a random variable.

**Example** If $k_0 = 1 - l_0$, $X$ is uniformly distributed on $[0, 1]$.

Indeed, $P(k_n = k, l_n = l) = \frac{1}{k + l - 1}$

$P(k_n = k, l_n = l) = \frac{1}{k + l - 2 \left[ \frac{k - 1}{k + l - 1} + \frac{l - 1}{k + l - 1} \right]} = \frac{1}{k + l - 1}$

What happens in general, if one starts at $(k_0, l_0)$?

**Theorem** Conditioned on $(k_0, l_0)$, $X_n = \frac{K_n}{K_n + L_n}$ converges to a Beta distributed random variable (having density

$$f(x) = \frac{(k_0 + l_0 - 1)!}{(k_0 - 1)! l_0 - 1} (1-x)^{k_0 - 1}$$

(Con can be proved by induction — somewhat messy).
Generalization: \( C \) colors. In this case, the limit fractions of balls of each color are uniformly drawn from the unit simplex \( \sum_{k=1}^{C} t_k = 1 \).

When \( C \) is large, Simon-Pareto distribution emerges: the frequency of \( k \)-th least frequent color behave as \( \frac{2k}{C^2} \)

Easiest proof: \( \mathbb{P} \sim \text{vector of iid } \mathcal{E}(1) \) variables normalized by sum.

Yule-Simon "Rich get richer" mechanism.
Another generalization: adding different amounts (even not integers!)

\[ R_n, B_n ; \begin{cases} R \text{ is drawn} & R_{n+1} = R_n + A_{rr} \\ B \text{ is drawn} & B_{n+1} = B_n + A_{bb} \end{cases} \]

Polya urn: \( A = (1, 0) \)

What happens for, say, \( A = (2, 1) \)?

Answer: \( X_n = \frac{R_n}{B_n + R_n} \to \frac{1}{2} \text{ a.s.} \)

More generally,

**Theorem** If all entries in \( A \) are positive, the \( X_n \) converges
d\( d = \frac{u_1}{u_1 + u_2} \), where \((u_1)\) is the unique eigenvector of \( A \) with positive coordinates (existing thanks to Frobenius-Perron).
For Polya urn case, any vector is eigenvector for $A_2(0^1)$. 

Heuristic explanation: take, for simplicity, the balanced case, where total weight added at each step is constant. Then $X_n = \frac{R_n}{T_n}$, and $R_n = R_{n+1} - R_n; T_n = T_{n+1} - T$

$$X_{n+1} - X_n = \frac{R_{n+1} + R_n}{T_n + T_{n+1}} - \frac{R_n}{T_n} = \frac{T_n}{T_n + T_{n+1}} X_n = \frac{1}{n} \left( \bar{F}(X_n) + E_{n+1} + R_n \right),$$

where $E(E_{n+1} \mid X_n) = 0$

and $|R_n|$ is not too large ($\Sigma |R_n|/n < \infty$ a.s.)

Then some relatively easy results show that for any $[a, b]$

where $\int_a^b |F(x)| > 0, P(X_n \in [a, b]) \to 0$.

This is the "stochastic approximation" regime.
Back to Polya urns, 
(Here $A=(1,0)$ and $X$ is a marking all, so that $F(x) = 0$).
Consider any trajectory $(k_0, l_0), (k_1, l_1), \ldots, (k_n, l_n)$, consisting of jumps $J_1, \ldots, J_n$.

**Proposition** The sequence $(J_1, \ldots, J_n)$ is exchangeable, i.e., that is

$P(J) = P(J_\sigma)$ for any permutation $\sigma$.

**Proof**: Direct check:

$P(J) = \frac{k_0 \cdot (k_0+1) \cdots (k_{n-1}) \cdot l_0 \cdots (l_n-1)}{(k_0+l_0) \cdots (k_n+l_n-1)}$

Exchangeable sequences have dramatic properties:
They (define) If $J_1, \ldots, J_k, \ldots$ is an infinite exchangeable sequence, then it is a mixture — Bernoulli $(p)$ sequences w.r.t. some measure $\mu$ on $[0, 1]$.

Proof is around the fact that $\frac{\# (J_k = R, k \in \mathbb{N})}{N}$ is a martingale.

It is worth noticing that finitely exchangeable sequences are not usually, mixtures, but are getting closer to them if they can be extended.

Example Consider exchangeable sequences of length 2.

$RR$ \quad $RB$ \quad $BR$ \quad $BB$ 

$a$ \quad $p$ \quad $p$ \quad $\gamma$

$a + 2p + \gamma = 1$
$d=\phi=0$, $\mu=\frac{1}{2}$ is exchangeable, but not extendable to length 3 (BBB BRR BRR BBB).

$\alpha \approx \mu \approx 0$

$x = z + \sqrt{3}$  $\beta = \frac{(a+b)}{3}$  $y = y + \frac{1}{2}$

As the distribution becomes extendable to longer sequences, it approaches blue region.

Another appearance of exchangeable sequences — in Edge Reinforced Random Walks.