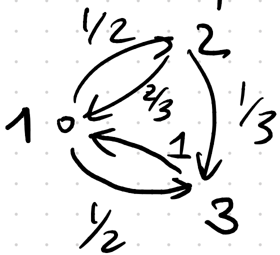
Spanning trees

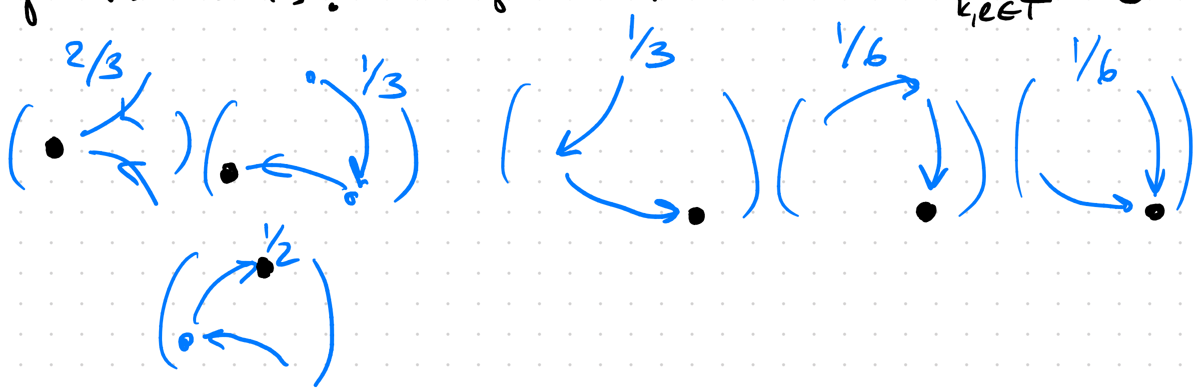
There are several deep connections btw random walks and spanning trees.

Let S, P — be Markov chain with finite state space S.

We describe it as a directed graph with weights P_{kl} $\underset{k}{\circ} \to \underset{l}{\circ}$

Spanning rooted tree is an oriented subtree with all arrows pointing to roots. Weight of a tree $= \prod_{k,l \in T} P_{kl}$

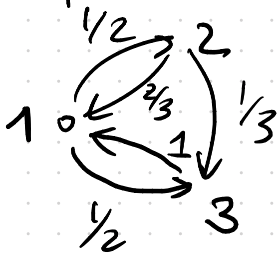




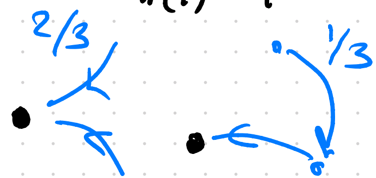
Thm (Markov chain tree) Invariant probability measure is proportional to sum of weights of trees rooted at k,

$$\pi(k) \sim \sum_{T \to k} w(T)$$

Example:

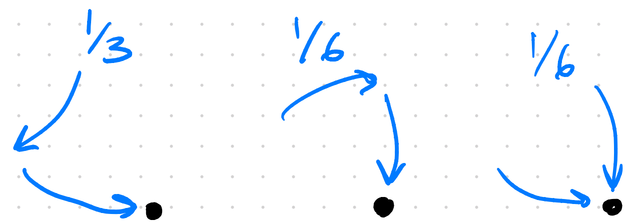


$\pi(1) \sim 1$

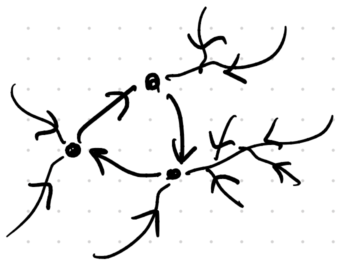


$\pi(2) \sim 1/2$

$\pi(3) \sim 2/3$



Proof Let $\{T \to k\}$ be the set of trees rooted at k; $\{Z\}$ – the set of rooted cycles; $Z^\circ \subset Z$ – the actual cycle. $w(G) = \prod_{k,l \in E(G)} p_{kl}$.



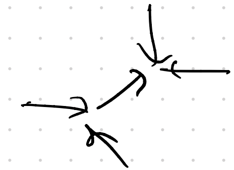
Then $\pi(k) = \sum_{T \to k} w(T) = \sum_{\substack{T \to k \\ l \in S}} w(T) \cdot p_{kl} =$

$$= \sum_{Z : Z^\circ \ni k} w(Z) = \sum_{m \in S} \sum_{\substack{Z : \\ (mk) \in Z^\circ}} w(Z) = \sum_{m} p_{mk} \cdot \sum_{T \to m} w(T) =$$

$$= \sum_{m} \pi(m) \cdot p_{mk}$$

$\square$

Matrix-tree theorem



As we saw, the rooted spanning trees give the invariant measure

$\pi(v) = \sum_{T \to v} w(T)$. If the chain is reversible,

$$p_{kl} = \frac{c(k,l)}{\pi(k)}, \quad \text{and} \quad w(T \to v) = \prod_{(kl) \in T} \frac{c(k,l)}{\pi(k)} =$$

$$= \frac{\prod_{e \in T} c(e)}{\prod_{k \in T} \pi(k)} \cdot \pi(v) = C \cdot w(T) \cdot \pi(v)$$

function of tree only

So, the key problem becomes finding normalization constant $\sum_{\substack{\text{spanning} \\ \text{trees of } G}} w(T)$, $w(T) = \prod_{e \in T} c(e)$

Let $\mathcal{S}$ be a finite network with weights $c(k,l)$ attached to its edges (everything's symmetric).

We call the matrix

$$(L)_{k,l} = \begin{cases} -c(k,l) & \text{if } k \neq l \\ +\sum\limits_{k\neq m} c(k,m) & \text{if } k = l \end{cases}$$

the Laplacian (we encountered it earlier as the energy, and called it Dirichlet kernel).

Easy to see that

$$(Lu,u) = \sum L_{k,l} \cdot u_k \cdot \bar{u}_l = \sum_{(k,l)} c(k,l)\,|u_k - u_l|^2 .$$

In particular, $L \geq 0$, and $\dim \operatorname{Ker} L = \#$ of connected components of the network.

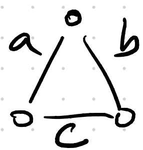
So, here's the remarkable

__Thm__ (Matrix tree thm, Kirchhoff '1847): If S is irreducible, the sum of weights of (unrooted) spanning trees

$$\sum_{T\text{-spanning}} w(T) = \det\left(L^{(mm)}\right), \text{ where } L^{(mm)} \text{ is (any)}$$

principal minor of L.

__Examples__ ①



$$\sum w(T) = ab + bc + ca$$

$$L = \begin{vmatrix} a+b & -a & -b \\ -a & a+c & -c \\ -b & -c & b+c \end{vmatrix} \qquad \det L^{11} = (a+c)(b+c) - c^2 \ \checkmark$$

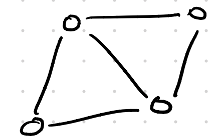
② If $c(k,l) \equiv 1$, we have enumeration of trees in a complete graph:

$$\#T(K_n) = \det\begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & & & -1 \\ \vdots & & & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix} = \det\left(nI_{n-1} - 1^T \otimes 1\right). \quad \sigma(-1^T \otimes 1) = \{-(n-1), \overbrace{0, 0, \ldots 0}^{n-2}\}$$

$$\sigma(nI_{n-1} - 1^T \otimes 1) = \{1, \underbrace{n, n, \ldots n}_{n-2}\} \implies \det\left(nI_{n-1} - 1^T \otimes 1\right) = n^{n-2} \ \text{((Cayley}$$

The proof, remarkably, involves some topology & some linear algebra.

Fix some orientations of the edges and



consider a version of the coboundary operator

$D: V_1 \to V_0$: $D: \delta(k,\ell) \longmapsto \rho_{k\ell}\cdot(\delta_k - \delta_\ell)$ $\qquad \rho_{k\ell} = [C(k,\ell)]^{1/2}$

Then $D^*: V_0 \to V_1$ is $D^* \delta_k = \sum_\ell \rho_{k\ell}\,(k,\ell)$

(a weighted analog of star operation) and, most importantly,

$$DD^* = L.$$

Take subset A of edges and subset B of vertices, $|A| = |B| = |S| - 1$.

Denote by D^A_B corresponding minor of D

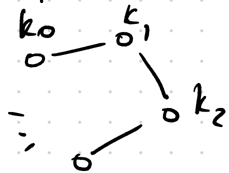
<u>Lemma</u> If graph defined by A is a tree, its det is $\pm \prod_{(k,\ell)\in A} \rho_{k\ell}$;

If it's not a tree, $\det = 0$.

Lemma If graph defined by A is a tree, its det is $\pm \prod_{(k,l)\in A} \rho_{kl}$;

If it's not a tree, det $= 0$.

Proof If on $|S|$ vertices a graph with $|S|-1$ edges from A isn't a tree, there should be a cycle



$k_n = k_0 - k_1 - \dots - k_{n-1} - k_n$

Then $\sum_{l=0}^{n-1} \rho^{-1}_{k_l k_{l+1}} (k_l k_{l+1})$ is in the kernel of D^A_B.

If the graph is a tree, choose the vertex in $|S| - B$ as a root, and number edges of A (and vertices of B) away from it. In this ordering, D^A_B is upper triangular, with ρ_{kl}, $(kl) \in A$ on the diagonal.

Last remarkable identity we need, is Cauchy-Binet formula

<u>Proposition</u> If A is $n \times m$, B is $m \times n$, then

$$\det AB = \sum_{\substack{I \subset \{1 \dots m\} \\ |I| = n}} \det A^I \cdot \det B_I \, , \quad A^I = \text{minor of } A \text{ with}$$

colums from I, B_I = minor of B with rows from I.

<u>Example</u> $a \cdot b = \sum a_k b_k$

Finishing Matrix-free theorem:

matrices without m-th row, column

$$\det L^{mm} = \det(DD^*)^{mm} = \det(D^{\hat{m}} D^*_{\hat{m}}) =$$

$$= \sum_{|I| = |S|-1} (\det D^{\hat{m}}_I)^2 = \sum_{T\text{-spanning tree}} \varphi^2(T) = \sum_T w(T)$$

$\square$