General Jump Processes

S - countable set; Q - matrix with $Q_{ii} = 0$; off-diagonal elements $\geq 0$.

Jump matrix: $\Pi_{kk} = \begin{cases} \frac{q_{kk}}{q_{kk}} & \text{for } k \neq l \\ 0 & \text{for } k = l \\ 1 & \text{for } k \neq l & \text{if } q_{kk} > 0 \text{ (absorbing state)} \end{cases}$

Example:

\[ \Pi = \begin{pmatrix} 0 & \frac{3}{8} & \frac{5}{8} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix} \]
So starting w Q matrix defining a \textit{continuous \ Markov process} valued in S

- initial distribution

Definition of continuous-time \textbf{MC} Markov(w, Q),

- run discrete time MC \( Y_n \sim \text{Mark}(\lambda_i, \Pi_i) \)

- set holding times \( S_n \) for \( X_n = k \) to be \( E(Q_{kk}) \)

\( \longrightarrow \) independent

- \( t > \sum_{n=1}^{\infty} S_n \), set \( X_t = \infty \) (special state)

- otherwise, for \( J_n \leq t < J_{n+1} \), \( X_t = Y_n \)
Two other ways to construct our jump processes from \( Q \) (the resulting process is equivalent to the original one)

Start w distribution \( X \) on \( S \), and array of iid \( G(1) \) variables \( T^k(u), k \in S; u = 1, 2, \ldots \). Then, if \( J_u = t, Y_u = k \).

1. Form \( S^k(u+1) = T^k(u)/q_{kl} \), for \( k \neq l \).
2. \( S(u+1) = \inf \{ S^l(u+1) \} \) (can be \( \infty \) if all \( q_{kl} = 0 \)).
   \( Y_{u+1} = \arg \min \{ S^l(u+1) \} \) (or \( k \) if \( S(u+1) = \infty \)).
   \( J_{u+1} = t + S(u+1) \).

Last construction: Introduce iid Poisson processes \( N_{kl}(\cdot) \) of rate \( q_{kl} \) for each edge \((kl)\); and define jump times, states inductively:

\( J_0 = 0; \quad J_{u+1} = \inf \{ t > J_u : N_{kl}(t) > N_{kl}(J_u) \} \)

\( Y_{u+1} = \text{that } k \) (or \( \infty \)) for some \( l \neq k \).
Explosiveness: Recall that the explosion time $T = \sup T_n$ is the time by which so many jumps happen.

Thus, $T = \infty$ a.s. if

$\text{a)}$ $S$ is finite, or

$\text{b)}$ $\sup \lambda_k < \infty$, or

$c)$ $X_0 = k$, and $k$ is recurrent for jump chain $Y_n$.

Then $a \Rightarrow b$; $b$ obvious (as $\sum \frac{1}{q_{Y_n}} = \infty$).

For $c$, $\sum \frac{1}{q_{Y_n}} = \sum \frac{1}{q_{Y_n}} = \infty$, where $n_e$ is the infinite subsequence where $Y_n = k$. 
In general, explosion times are hard to detect. One general result here is

Thus for a continuous time Markov chain w. generator \( \mathcal{Q} \),

set \( \tau_k = E_k e^{-\Theta T} \) \((\Theta > 0, T\) - explosion time). Then \( \mathcal{Q} \tau = \Theta \tau \) \((\ast)\), hence, if \( \tau \) satisfies \((\ast)\) and

\[ |\tau_k| \leq 1 \text{ for all } k, \text{ then } \tau_k \leq \tau_1 \text{ for all } k. \]

**Proof.** Conditioned on \( X_0 = k \), the time and rate of 1st

jump are independent:

\[ P_k (Y_1 = e) = \pi_k e = \frac{q_k e}{q_k}; J_1 \sim E(q_k) \]

After the jump, we get again Markov chain at \( e \), w. generator \( \mathcal{Q} \).

\[ \mathbb{E}_k (e^{-\Theta T} / Y_1 = e) = \mathbb{E}_k (e^{-\Theta J_1 - \Theta (T - J_1)} / Y_1 = e) \]

\[ = \mathbb{E}_k e^{-\Theta J_1}. \mathbb{E}_k e^{-\Theta T} = \mathbb{E}_k e^{-\Theta J_1 \cdot \tau_2} = \frac{q_k \cdot \tau_2}{q_k + \Theta} \]
\[ Z_h = \sum_{l \in \mathcal{L}} \mathbb{E}(e^{-\theta T} | Y_l = l) \hat{p}_k(Y_l = l) = \sum_{l \in \mathcal{L}} \frac{q_k}{q_k + \theta} \cdot Z_l \cdot \frac{q_{kl}}{q_k} = \frac{1}{q_k + \theta} \sum_{l \neq k} q_{kl} \cdot Z_l \]

\[ q_k Z_k + \Theta Z_k = \sum_{l \neq k} q_{kl} Z_l \Rightarrow \Theta Z_k = \sum_{l \neq k} q_{kl} Z_l \quad \checkmark \]

Slightly expanding, we see that
\[ E_k e^{-\Theta J_{k+1}} = \sum_{l \neq k} E_k e^{-\Theta J_l} - \Theta \left( \sum_{m=2}^{\infty} \mathbb{E} \left[ \mathbb{E}_k e^{-\Theta J_m} \right] \right) \mathbb{P}(Y_i = l) = \]

\[ = \frac{1}{q_k + \theta} \sum_{l \neq k} q_{kl} \cdot E_l e^{-\Theta J_l} \]
So, if \( \tilde{x}_k \leq 1 \) for all \( k \), then
\[
\tilde{x}_k \leq \frac{q_k e^{-\theta T_n}}{q_k + \theta}, \quad \text{and if} \quad \tilde{x}_k \leq \frac{q_k e^{-\theta T_n}}{q_k + \theta},
\]
and
\[
\theta \tilde{x}_k = \sum q_k e \tilde{x}_k, \quad \text{then}
\]
\[
\tilde{x}_k = \sum \frac{q_k e \tilde{x}_k}{q_k + \theta} \leq \sum \frac{q_k e e^{-\theta T_n}}{q_k + \theta} = e^{-\theta T_n} e^{-\theta T_n} = e^{-\theta T_n},
\]
so
\[
\tilde{x}_k \leq \lim_{n \to \infty} e^{-\theta T_n} = e^{-\theta T} = 2_k
\]
dominated convergence.

Ripping \( \tilde{x}_k \) to \( -\tilde{x}_k \) proves the claim.

**Corollary**: \( \mathcal{Q} \) non-explosive \( \implies \) no bounded eigenvectors \( Q \).

**Proof**: Indeed, if \( \mathcal{Q} \) is explosive, \( \mathcal{Q} (T<\infty) > 0 \), so \( e^{-\theta T} > 0 \), and is a bounded eigenvector. If \( \mathcal{Q} \) is non-explosive, any bounded eigenvector \( \leq e^{-\theta T} = 0 \), hence 0.
\textbf{Backward & Forward equations}

Recall that for finite $S$ (and $\Omega$), the solutions to matrix differential equations

$$\dot{P} = Q P, \quad P(0) = I, \quad \dot{P} = P Q$$

exist and are given by

$$P = e^{Q t} = I + Q t + \frac{Q^2 t^2}{2} + ...$$

Notice that these are different systems.

\textbf{E.g. for birth chain:}

\[ 0, q_1, q_2, q_3, q_4, \ldots \]

$$\dot{P}_{ke}(t) = q_k P_{ke+1}(t) - q_k P_{ke}(t) \quad \left[ \text{Backward equation} \right]$$

$$\dot{P}_{ke}(t) = P_{ke-1}(t) \cdot q_{e-1} - P_{ke}(t) q_e \quad \left[ \text{Forward equation} \right]$$
The purpose of these equation is to split interval 
$[0, t+\epsilon]$ into $[0, \epsilon] \cup [\epsilon, t+\epsilon]$ (backward eqn) \n$[0, t] \cup [t, t+\epsilon]$, and use the Upper property.

The resulting family of operators $\mathcal{P}(t)$, $t \in \mathbb{R}$
forms a matrix group.

$$\mathcal{P}(t+s) = \mathcal{P}(t) \mathcal{P}(s)$$

If $\mathcal{P}$ is infinite, the exponent $e^{\sqrt{s}}$ is not
really defined in general, so one needs to be
careful.

Luckily, one still has the key theorem:

\underline{Theorem.} Given a $\mathcal{P}$ matrix, there exist a matrix

semi-group $\mathcal{P}(t)$, $t \geq 0$ [i.e. $\mathcal{P}(t+s) = \mathcal{P}(t) \mathcal{P}(s)$]
Solving both backward & forward equations

\[ \dot{\Phi} = \lambda \Phi; \quad \Phi(0) = I \quad \dot{\Phi} = PQ \]

such that \( \Phi \) is substochastic matrix, giving

the transition probabilities \( p(x; t) \) for the Wright

death with generation \( Q \). [This is the description for minimal sol’n]

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Worthwhile working out an example:

Take \( Q_{kl} = \begin{cases} -1 & \text{for } k = l \\ 1 & \text{for } k \neq l \text{ and } N \end{cases} \)

Then transition probabilities solve

\[ \dot{\Phi} = PQ \]. \text{ As usual, if } Q \text{ is diagonalizable}

with simple spectrum, then
\[ \lambda_e v_e = \Omega v_e, \quad e = 0, \ldots, n-1, \] then
\[ \lambda_e u_e = u_e \Omega \]
\[ e^{Q^T} = \sum e^{\lambda_e} v_e^T u_e = P(t). \]

Any cyclic matrix is diagonalizable (it commutes with representation of Abelian group \( \mathbb{Z}_N \)), and the eigenpairs are
\[ v^{(e)} = (v^{(e)}_0, \ldots, v^{(e)}_{n-1}) \quad e = 0, \ldots, n-1, \]
\[ v^{(e)}_k = \varepsilon^{ke}, \quad \text{where} \quad \varepsilon = e^{2\pi i/n} \]

In our case
\[ [Q v^{(e)}]_k = -v^{(e)}_k + v^{(e)}_{k-1} = \begin{cases} 0, & k = 0, \ldots, n-2 \\ 0, & k = n-1 \end{cases} \]
\[ = -\varepsilon^{e^k} + \varepsilon^{e^{(k-1)}} = (\varepsilon^{-e} - 1) \cdot v^{(e)}_k \]
\[ \lambda_e \]
To find $P_k(t)$, we take $k$-th basis vector $e_k$, expand in terms of eigenvectors: $e = \sum c_k \psi^{(k)}_i$

$$e_k = \sum c_k \psi^{(k)}_i = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if not} \end{cases} \Rightarrow c_k = \frac{1}{N} \quad \text{would work...}$$

So, $P_k(t) = \frac{1}{N} \sum_{k=0}^{N-1} e^{i(A + \frac{\varepsilon}{N} + \varepsilon)} = \frac{1}{N} \sum_{k=0}^{N-1} e^{\left(\varepsilon - 1\right)t} \varepsilon_k$

Take $N = 4$ for simplicity: $\varepsilon = i$

$$P_{00}(t) = \frac{1}{4} \left(1 + e^{(-i-1)t} + e^{-2t} + e^{(i-1)t}\right) = \frac{1}{4} + \frac{1}{4} \left(e^{-2t} + 2e^{-t} \cos t\right).$$

**Exercise**: What about torus?