Reversible Markov Chains

\((\xi, \mathbb{P})\) be an irreducible MC, \(\pi\) - invariant distribution.

\(X_n, n=0, \ldots, N\) - a (finite time) process with \(X_0 \sim \pi\).

Then \(Y_{n} := X_{n-1}\) is also Markov, for \(\tilde{\mathbb{P}}\) given by

\[\pi_e \tilde{P}_{ek} = \pi_k \mathbb{P}_{ke}\] (i.e. \(\tilde{P}_{ek} = \frac{\pi_k \mathbb{P}_{ke}}{\pi_e}\))

Indeed, \(\mathbb{P}\left(X_0 = e_0, \ldots, X_N = e_N\right) = \pi_{e_0} \mathbb{P}_{e_0 e_1} \cdots \pi_{e_N} \mathbb{P}_{e_N e_0} = \pi_{e_0} \tilde{P}_{e_0 e_1} \pi_{e_1} \tilde{P}_{e_1 e_2} \cdots \pi_{e_N} \tilde{P}_{e_N e_0} = \pi_{e_0} \tilde{P}_{e_0 e_1} \cdots \pi_{e_N} \tilde{P}_{e_N e_0} \overline{e_N} = \pi_{e_0} \tilde{P}_{e_0 e_1} \cdots \pi_{e_N} \tilde{P}_{e_N e_0} \overline{e_N} = \mathbb{P}\left(Y_0 = e_0, Y_1 = e_1, \ldots\right)\)
Of course, it is immediate that $\tilde{P}$ is stochastic, and irreducible.

$(\tilde{u}, \tilde{P})$ is called time reversal of $P$.

Example

\[ \begin{array}{ccc}
1 & 1 & 2 \\
\downarrow & \downarrow & \downarrow \\
P & 1 & \tilde{u} = \left( \frac{1}{12}, \frac{1}{12}, \frac{10}{12} \right) \\
\downarrow & \downarrow & \downarrow \\
0.9 & 0.1 & 1 \\
\end{array} \]

\[ T_1 \tilde{P} \overset{\sim}{=} \tilde{w}_3 P_{21} \]

\[ T_2 \tilde{P} \overset{\sim}{=} \tilde{w}_2 P_{23} \]
If $\pi$ is a measure on $S$, and

$$\pi(k) = \pi(e) \pi(k)$$

for all $k, e \in S$. (DB)

We say that $\pi, \rho$ are in detailed balance.

NB: $\pi, \rho$ in detailed balance $\Rightarrow \pi \rho = \pi$

We say that $\pi$ is reversible if $\pi$ satisfying (DB) exists.

NB: $\pi$ not necessarily finite: if it is, the stationary measure $\pi$ for $\pi$ exists, and in equilibrium,

$$\pi = \pi X$$

is equivalent (in distribution) to $X$.  

Detailed balance
Examples:

\[ \lambda_k \rho = \lambda_{k+1} \cdot q \]

\[ \lambda_{k+1} = (\frac{p}{q}) \lambda_k \]

\[ \lambda_k \sim \lambda_k \quad p < q \]

Random walks on a graph: from any vertex, jump at random to any of the neighbors.

I.e. \( P_{ke} = \frac{1}{v(k)} \) for \( k \neq l \)

Slight generalization:

\[ P_{ke} = \frac{w_{ke}}{\sum w_{ke}} \]

\[ \lambda_k = \sum w_{ke} \]
Big question: when PC is reversible?
When all coboundaries are 0!

\[ \Pi P_{k+t_{k+1}} = \Pi P_{k_{k+1}, k_t} \]
(Enough to consider only cycles generating \( H_1 \) of the graph)

This is the best part about reversible Markov chains: to compute (multiples of) stationary distribution, computing detailed balance along some spanning tree is enough.

Example: Random walks on regular graphs: stationary probability - uniform.
Jumping together

A lattice chain: two particles jump independently, uniformly to a neighboring site. How long one needs to wait until they are all back?

\[ \tilde{w}_E = \frac{1}{\tilde{u}_E} \Rightarrow \tilde{w}_k = ? \]