**Q matrices & exponentials**

$S$ - countable, $Q = (q_{kl})_{k \in S}$.
- $-\infty < q_{kl} \leq 0$
- $q_{kl} > 0$ if $l \neq k$
- $\sum q_{kl} = 0$ (or, $Q \mathbf{1} = 0$).

$q_{kl}$ - rate of escape to $l$ from $k$; $q_{kk}$ - total escape from $k$ rate.

One can think of $Q$ as "log" of stochastic matrices.

Let $|s| < \infty$ for now.

$e^{tQ} = \mathbf{1} + tQ + \frac{t^2 Q^2}{2!} + \cdots + \frac{t^n Q^n}{n!} + \cdots$

Then $e^Q$ is well-defined (via power series), and $e^{Q_1 + Q_2} = e^{Q_1} \cdot e^{Q_2}$ if $Q_1, Q_2$ commute. ($Q_1 Q_2 = Q_2 Q_1$)

Also, if $P(t) = e^{tQ}$, then

\[ P(0) = \mathbf{E} \]

\[ \frac{dP}{dt}(s) = P(s) \quad Q = Q P(s) \quad \frac{dP}{dt + 2}(s) = Q^2 P(s) \]

\[ P(t) = e^{tQ} \]

\[ Q \mathbf{1} = 0 \]
Need to prove that $e^{\Theta}$ makes sense; the rest is formal computations.

If $|\Theta|$ is any operator norm — that is $|A+B| \leq |A|+|B|$ (e.g., $\sup_{v \neq 0} \frac{|\Theta v|}{|v|} \Rightarrow \sum \Theta^k/k!$ converges $|AB| \leq |A||B|$)

(partial sums form Cauchy sequence in $\text{Mat}(\mathbb{R}^{m\times n})$)

Relation to stochastic matrices

On finite set $S$, $Q$ is $Q$-mech $\iff P(t) = e^{t\Theta}$ is stochastic for all $t$.

$\Rightarrow e^{t\Theta} 1 = 1 + \sum_{k \geq 1} \Theta^k/k! 1 = 1$. Also $e^{t\Theta} = \lim_{n \to \infty} (I + \frac{t\Theta}{n})^n$

so for $n$ large enough $(I + \frac{t\Theta}{n})$ has positive coefficients and so does $P(t)$.

Hence, $e^{t\Theta}$ is stochastic for any $t$. 

Proof
\[ \text{If } e^{tQ} = 1 \text{ for all } t, \text{ then } 0 = \frac{d}{dt} e^{tQ} \bigg|_{t=0} = Q. \]

So, embedded into ROZ is a collection of stochastic matrix transients.

**Example** \( Q = \begin{pmatrix} -2 & 1 & 1 \\ 2 & -3 & 1 \\ 1 & 0 & -1 \end{pmatrix} \)

\[
e^{tQ} = \begin{pmatrix} \frac{3}{8} + \frac{1}{4} e^{-2t} + \frac{1}{8} e^{2t} & \frac{1}{8} e^{-2t} + \frac{1}{2} e^{2t} & \frac{1}{2} e^{-2t} - \frac{1}{2} e^{2t} \\ 3/8 + \frac{1}{4} e^{-2t} + \frac{1}{2} e^{2t} & \frac{1}{8} e^{-2t} + \frac{1}{2} e^{2t} & \frac{1}{2} e^{-2t} + \frac{1}{2} e^{2t} \\ 3/8 + \frac{1}{2} e^{2t} & \frac{1}{8} e^{2t} & \frac{1}{2} e^{2t} \end{pmatrix}
\]

\[ A^3 + 6A^2 + 8A = \lambda (A+2)(A+4) \]

**Example 6** \( \begin{array}{c} A \rightarrow 0 \rightarrow A \rightarrow 0 \rightarrow A \rightarrow 0 \rightarrow \end{array} \quad Q = \begin{pmatrix} -a & a & 0 \\ -a & a & 0 \\ 0 & 0 & -a \end{pmatrix} = \frac{\lambda}{(-E+N)} \)

\[
A^3 + 6A^2 + 8A = \lambda (A+2)(A+4)
\]
Augmenting it with last row & columns (which is necessarily equal to \((0, 0, 0, \ldots, 0, 1)\)) gives the stochastic matrix.

Not all stochastic matrices are matrix exponentials of Markov matrices:

**Example:** there is no \( Q \): \( e^Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \)

\( \text{Prop. det } e^Q = e^{trQ} \) \( e^{trQ} \neq -1 \)
Returning to clean — the transition \[ \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \ldots \]

Probabilities \[ P_{o_k}(t) = e^{-\lambda t} \frac{(\lambda t)^k}{k!} \] from (fraction of)

Poisson distribution w parameter \( \lambda t \)

(Recall that for \( X \sim \text{Poisson}(\mu) \), \( P(X=k) = e^{-\mu} \frac{\mu^k}{k!} \) )
Random processes on $\mathbb{R}$

Random process on $\mathbb{Z}$ is easy — just a sequence of r.v. What about processes on $\mathbb{R}$?

Much harder to reason about trajectories, taking into account continua of values (for which usual axioms of measure are not really working).

Here looking only at right-continuous (even right-constant) processes:

For any $t, \omega$, there is $\varepsilon$:

$$X_{t+\varepsilon}(\omega) = X_t(\omega) \text{ for } 0 \leq \varepsilon \leq \varepsilon(\omega)$$

Then measure on such processes is fixed when its finite-dimensional distributions are fixed.

(\text{e.g. } \{ \sup_t X_t < a \} \iff \bigcap_{t \in \mathbb{Q}} \{ X_t < a \})
What pathologies a (random) trajectory can have?

Consider \( \{ t \in \mathbb{R} : \lim_{s \to t} X_s \neq X_t \} \) — the set of “jumps.”

Then

a) the set of jumps is well-ordered in \( \mathbb{R}_{\geq 0} \) (no infinite decreasing sequences of jumps)

b) One can have set of jumps
   - finite (get stuck in a point forever)
   - locally finite (each interval in \( \mathbb{R} \) contains only finitely many jumps)
   - accumulate to a value on the right: as many jumps over finite interval.

\( J \alpha \) — jump times;

\( S \alpha = J_{\alpha+1} - J \alpha \) — holding time; \( \sum S_n = \tau \) — explosion time.

The process till explosion is \text{minimal} process.