Convergence to equilibrium

We saw examples already: \( P^n \rightarrow \) rank 1 matrix

\[
A \cdot \pi = \begin{pmatrix}
\pi_1 & \pi_2 & \cdots
\end{pmatrix}
\]

Easy to see it happen when \( P \) has spectrum

\[
\sigma(P) = \{1, \lambda_2, \ldots \} \quad \text{and} \quad |\lambda_k| < 1.
\]

[This is the linear-algebra, via Frobenius-Perron theorem]

We use probabilistic route.

First: problems that can arise:

\[
\pi P = \pi \Rightarrow \pi = (\frac{1}{2}, \frac{1}{2}).
\]

\[
P = \begin{cases}
(0, 1) & \text{if odd} \\
(0, 0) & \text{if even}
\end{cases}
\]

\[
\pi^n = \begin{pmatrix}
\pi_1^n & \pi_2^n
\end{pmatrix}
\]

\[
\begin{pmatrix}
\pi_1 & \pi_2
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} =
\begin{pmatrix}
\pi_1 & \pi_2
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{1}{2} & -\frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} = 
\begin{pmatrix}
\frac{1}{2} & -\frac{1}{2}
\end{pmatrix}
\]
Period of a state = GCD (lengths of all paths in \( S \) through that state with edges \((k,e)\): \( P_{ue} > 0 \))

State \( k \) is aperiodic if its period = 1 (equivalently, \( P^m_{kk} > 0 \) for all \( n \) large enough. (Change problem)

For an irreducible \( P \), one state is aperiodic \(\implies\) all states are.

Proof: the usual

\[ P^n_{ab} > 0 \quad P^{n+1}_{ab} > 0 \]

\( P \) — irreducible, aperiodic & \( \pi \) is an invariant distribution. Then \( P^n \to A \cdot \pi \) (in particular, \( A P^n \to \pi \), no matter what \( A \) is)
(N3: convergence is element-wise ⇒ in $l^0$ norm).

Equivalently, distribution of $X_n : X_n = k \Rightarrow \mathbb{0}$ for any $k$.

Proof: by coupling.

Let $Y_n = 	ext{Mirror}(\bar{a}, P)$, independent of $X$.

Then the pair $(X_n, Y_n)$ is a Mirror chain on $S \times S$.
(The initial state is, e.g. $(S_0, \bar{a})$; trans. matrix $P \otimes P$)

Fix a state $l$, and wait till $X_n = Y_n = l$ (i.e. take $T = \min \{ n : X_n = Y_n = l \}$) and
swap $X$ & $Y$ after $T$:
\[(X_{u}', Y_{u}') = \begin{cases} (X_{u}, Y_{u}) & \text{for } u \leq T \\ (Y_{u}, X_{u}) & \text{for } u > T. \end{cases} \]

Strong Markov property \(\Rightarrow\) \(X_{u}', Y_{u}'\) is again a Markov process with the same initial prob. & transition matrix \(P \otimes P\).

I.e., the distribution of \(X_{u}'\) is the same as the distribution of \(X_{u}\).

So, \[|P(X_{u} = w) - \overline{u}_{w}| =
\begin{align*}
&= |P(X_{u}' = w) - \overline{u}_{w}| = |P(X_{u} = w, u < T) + P(Y_{u} = w, u \geq T) \\
&\quad - P(Y_{u} = w, u < T) - P(Y_{u} = w, u \geq T)| \\
&\leq P(u < T).
\]
Lastly, \((X_n, Y_n)\) has a stationary prob. distribution \(\pi \times \pi\). So, it is recurrent and \(P(T<\infty)=1\)
\[\implies P(T \geq n) \to 0 \text{ as } n \to \infty\]

Hence, \(\sum_k P^n \to \pi\) (pointwise)

Where did we use aperiodicity?
Where we asserted that \((X_n, Y_n)\) is irreducible: If \(P\) has a period, then all states split into classes \(k \mod P, \ k=0, \ldots, P-1\), and \(X_n, Y_n\) starting at different classes could never reach \((l, l)\).
Before is $E = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Then $P = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$. Let $\pi = P \in \mathcal{P}(E)$. Assume that $P$ is connected by paths of positive probability. Then $P$ is aperiodic. The general convergence theorem is applicable. Let $\mathcal{E}$ be the set of positive probability classes of $P$. For $\pi \in \mathcal{E}$, the convergence to equilibrium is automatic. You just observe its behavior at multiples of $\pi$ — then it is periodic.
Example: Let $S = S_m$, the group of permutations of $m$ elements. At each step, one swaps 2 neighboring elements. Then a P1 converges to $\lambda_0 U(A) + \lambda_1 (U(\sigma_{12} A))$, where $U(A)$ is the uniform distribution on the group of even permutations ("alternating group"); $\sigma_{12} A$ is the class of odd permutations.

\[
\begin{align*}
\frac{1}{m-1} & \quad \frac{1}{m-1} \\
& \quad \quad \vdots \\
\frac{1}{m-1} & \quad \frac{1}{m-1}
\end{align*}
\]
$k$ is aperodic if for any $N > N_0$, there is a (positive prob.) trajectory $k \to k$ of length $N$. 

\begin{tikzpicture}[baseline=(current bounding box.center)]
  
  \node (a) at (0,0) {1};
  \node (b) at (1,0) {2};
  \node (k) at (2,-1) {$k$};
  \node (q) at (4,0) {q};
  \node (t) at (5,0) {t};
  \node (b) at (3,-2) {$b$};

  \draw[-stealth] (a) to[bend right] (b);
  \draw[-stealth] (b) to[bend left] (a);
  \draw[-stealth] (k) to[bend right] (q);
  \draw[-stealth] (q) to[bend left] (k);
\end{tikzpicture}