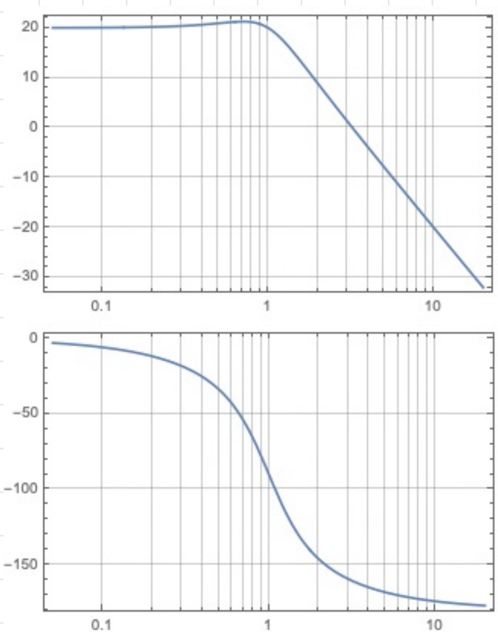
Nyquist Plots, Nyquist criterion

Reading: FPE 6.3

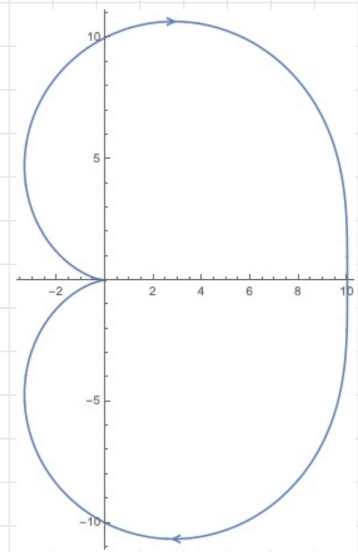
Bode plots vs Nyquist plots

Both are plots of the transfer function restricted to imaginary axis (frequency response).

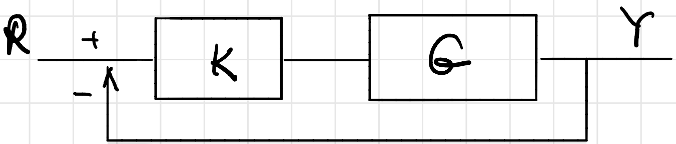
Insignificant difference: Bode plots shows the log of transfer function
Significant difference: Bode plots a pair of functions; Nyquist plot is parametric, - easier to understand the global behavior.



$$G = \frac{10}{s^2+s+1}$$



Particular usage of Nyquist plots: determine the number of RHP poles of the closed loop transfer function



$$\frac{Y}{R} = \frac{KG}{1+KG}$$

The poles of the closed loop transfer function correspond to the zeros of *1+KG*, or values where

G+1/K=0

In other words, we can argue entirely about the open loop transfer function…

Enter Nyquist plots

Assume: $G = \frac{p}{q}$ $\quad \deg p \leq \deg q$

Interested in

$$\frac{KG}{1+KG} = \frac{Kp}{q+Kp} \leftarrow \text{poles}$$

$\Updownarrow$

$$1+KG = \frac{q+Kp}{q} \leftarrow \text{zeros}$$

$\Updownarrow$

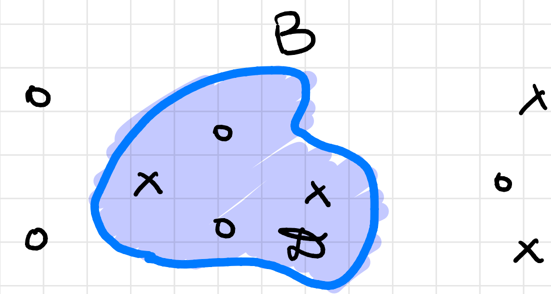
$$G + 1/K = 0 \qquad \leftarrow \text{zeros}$$

Where $G = -1/K$?

Consider first the following question:
Given a rational function

$m \le n$ (proper transfer fn!)

$$H(s) = \frac{(s-z_1)\dots(s-z_m)}{(s-p_1)\dots(s-p_n)}$$



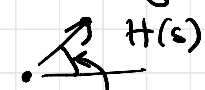
and a domain in the complex plane,
how many zeros it has in that domain?

$$\#\{ s : H(s) = 0 ,\ s \in D \}$$

Preferably, we would like to examine only
the points on the boundary of that domain.

We cannot do it, - but we can count the number of zeros less the number of poles in that domain, - by tracking the argument of our function as we track the boundary of the domain

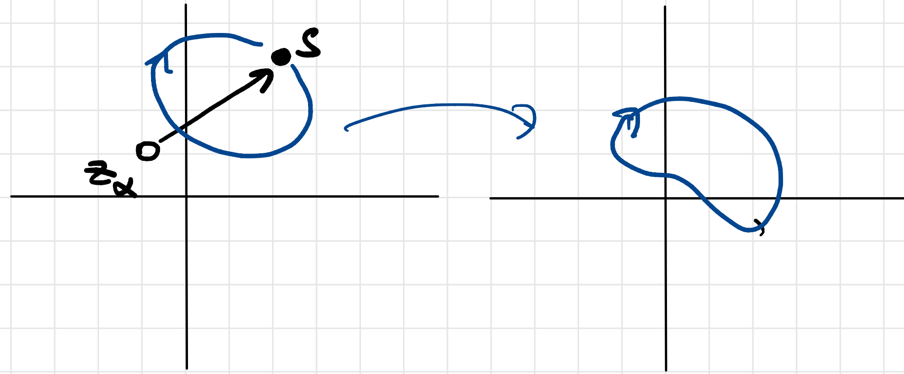
$$s \in B = \partial D \rightsquigarrow H(s) = e^{M(s)} \cdot e^{j \operatorname{Arg}(H(s))} \qquad M = \log |H(s)|$$



$$\operatorname{Log} H(s) = M(s) + j \underbrace{\left(\sum_{\alpha=1}^{m} \operatorname{Arg}(s - z_\alpha) - \sum_{\beta=1}^{n} \operatorname{Arg}(s - p_\beta) \right)}_{\operatorname{Arg} H(s)}$$

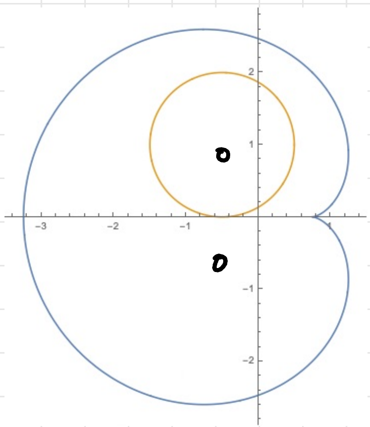
The fact that the logarithm of the product is the sum of logs of the factors implies we have to track only the logs of those factors.

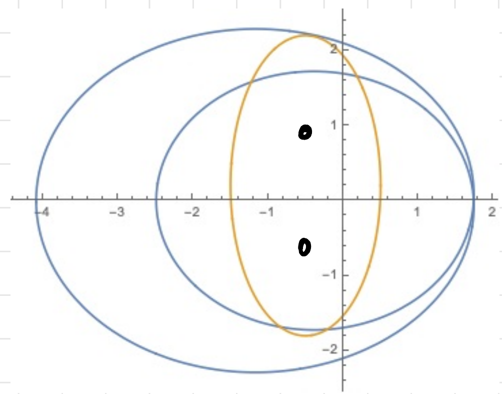
If the zero is inside the domain, the argument moves around the origin once clockwise, as *s* moves around the boundary of the domain clockwise.

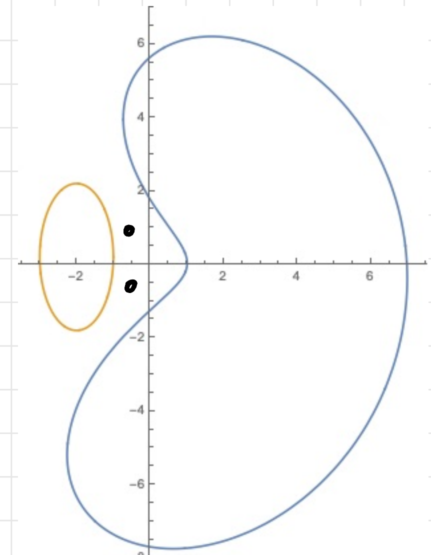


If the zero is outside the domain, the argument does not encircle the origin at all.

$H = s^2 + s + 1$





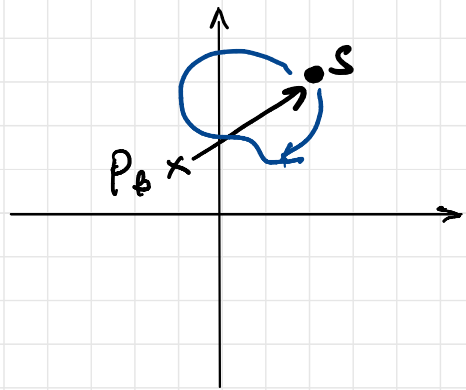


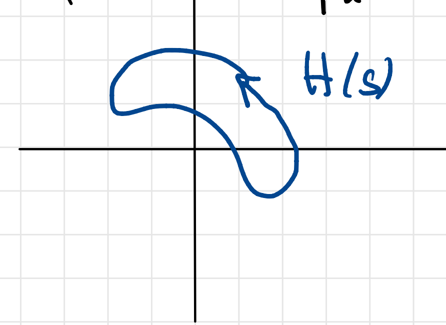
Of course, the situation with the factors in the denominator of the rational function is completely symmetric. The only difference is that the argument changes the direction in which it circumvents the origin: it now goes counterclockwise.

$$\operatorname{Arg}\frac{1}{H} = -\operatorname{Arg} H$$

Arg $H(s)$ -- 2π for each $p_\alpha \in \mathcal{D}$

If the pole is inside the domain, the argument moves around the origin once *counter-*clockwise, as *s* moves around the boundary of the domain clockwise.



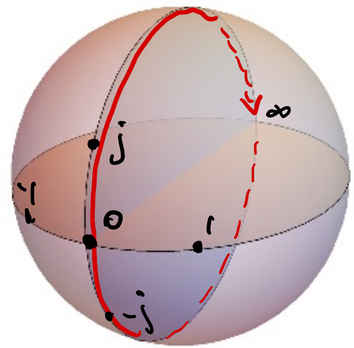


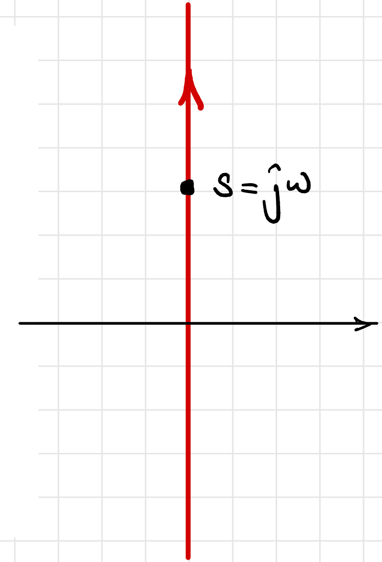
As before, if the pole is outside the domain, the argument does not encircle the origin at all.

$$H(s) = \frac{(s-z_1)\dots(s-z_m)}{(s-p_1)\dots(s-p_n)}$$

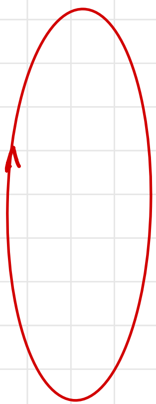
To summarize: the image of the (clockwise-traversed) boundary under a rational function circumvents the origin clockwise ***Z-P*** times, where *Z* is the number of zeros and *P* is the number of poles in the domain.

Back to our problems: the
domain we are interested in is
the right half-plane.
Will our approach work there?





Yes!
Because the imaginary axis
is a circle…



To find out how many zeros minus poles of
a rational function in the right
hemisphere are there, we can just count
how many times the contour obtained by
mapping the imaginary axis (run S->N)
surrounds the origin clockwise.

In our situation, we are looking at the number of RHP poles of

$$\frac{KG}{1+KG} = \frac{Kp}{q+Kp}$$

for $G = \frac{p}{q}$

This is equivalent to looking for number of zeros of the polynomial

$$q + Kp$$

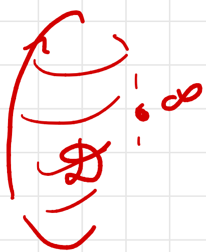
…and the number of zeros of

$$q + Kp \qquad (Z)$$

is the the difference of the number of zeros and the number of poles of

$$\frac{q+Kp}{q} = 1 + KG$$

…plus the number of poles of $\frac{p}{q}$ - the open loop transfer function!



So, if we know the number of poles of the open loop transfer function in the RHP,

$$(P)$$

and the number of times the image of the imaginary axis under

$$1+KG$$

surrounds (clockwise) the origin,

$$(N)$$

we are done

$$N = Z - P \qquad Z = N + P$$

But the latter number is the number of times the image of the imaginary axis under the open loop transfer function (i.e. the Nyquist plot) surrounds $-1/K$.

$$\# \; G + \frac{1}{K} \text{ surrounds } 0 = \# \; G \text{ surrounds } -1/K$$

Summary (Nyquist theorem)

If G is a rational function with no poles or solutions of $1+GK=0$ on the imaginary axis, then

N, the (clockwise) winding of the Nyquist plot around -1/K=

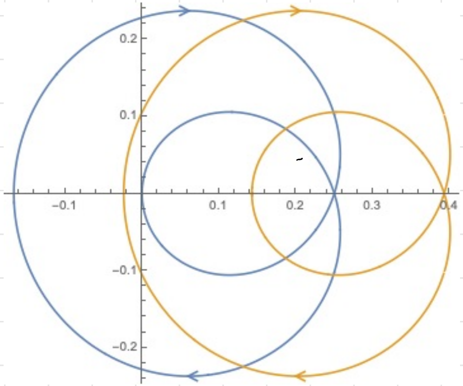
#(zeros of 1+KG in the RHP)-
#(poles of 1+KG in the RHP)=

$Z-P$ =

#(closed loop transfer function poles in the RHP)-
#(open loop transfer function poles in the RHP)

The number we are interested in is Z - the number of poles of the closed loop transfer function in the RHP.

The number of windings is obtained from the Nyquist plot.
The number P is independent of K.



$$Z = P + N$$

Example: $G = \dfrac{s-1}{s^2+4s+6}$ $p = s-1$

$q = s^2+4s+6$

$\{G(j\omega)+1/7,\ \omega \in \mathbb{R}\}$

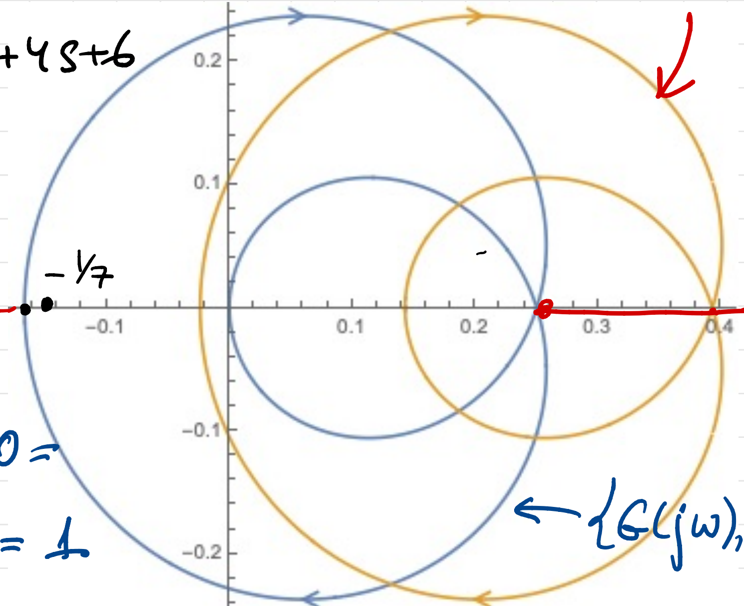
$P = \#$ roots of q in RHP $= 0$

$p_{1,2} = -2 \pm j\sqrt{2}$

$K = -1/7$ $G + 1/K$

$N = \#$ windings of $G + 1/K$ around $0 =$

$= \#$ windings of G around $-1/K = 1$



$\Rightarrow$ for $K=7$, $Z = N + P = 1$, K - unstable gain.

Routh: $s^2+4s+6+K(s-1) =$

$= s^2 + (4+K)s + (6-K)$

Stability $\Leftrightarrow$ $4+K \geq 0$ $-4 \leq K \leq 6$

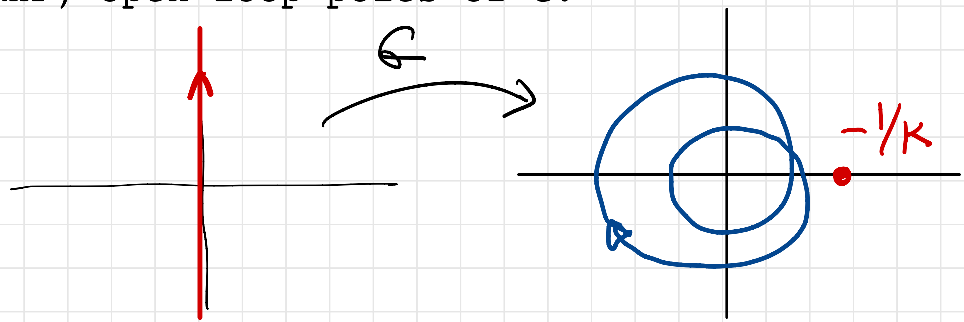
$-1/K \leq -1/6$ or

$-1/K \geq 1/4$

Nyquist Stability Criterion:

Under the assumptions of the Nyquist theorem, the closed-loop system (at a given gain K) is stable if and only if the Nyquist plot of G encircles the point $-1/K$ counterclockwise, as many times as there are unstable (RHP) open-loop poles of G.

Stability <=> *N+P=0*



We can either sketch Nyquist plot (directly or from from Bode plots), or work with Bode plots.

Advantages of Nyquist over Routh–Hurwitz:

- can work directly with experimental frequency response data (e.g., if we have the Bode plot based on measurements, but do not know the transfer function)
- aesthetically pleasing