Hitting times & absorption probabilities

For Markov $(A, P)$ process, often we want to know, where we land first, in some subset $A$ or $B$?

Standard example: drunkard's walk:

\[ A \xrightarrow{*} B \]

One can introduce absorption probability \((\text{for a closed class } A, \text{this is just } P_k(\text{hit } A) = h_k)\).

Also, one can ask how long would it take to reach $A$...

**Def.** Hitting time $\tau^A = \inf\{n \geq 0 : X_n \in A\}$.

Note: $\tau^A$ is a random variable — an example of stopping time...
So, we have \( h_k^A = \mathbb{P}_k(H^A < \infty) \), \( m_k^A = \mathbb{E}_k(H^A) \).

\[
= \sum_{n=1}^{\infty} \mathbb{P}_k(H^A = n) \leq \sum_{n=1}^{\infty} \mathbb{P}_k(H^A > n)
\]

(Or \( \infty \) if \( h_k^A < 1 \))

Example

\[ \begin{array}{c}
0 \to \frac{1}{2} \to 0 \to \frac{1}{2} \to 0 \to 3
\end{array} \]

\[ h_1^0 = \frac{1}{2} + \frac{1}{2} h_2^0 \quad h_2^0 = \frac{1}{2} h_1^0 + \frac{1}{2} h_3^0 \]

\[ h_1^0 = \frac{2}{3} \quad h_2^0 = \frac{1}{3} \quad h_0^0 = 1 \quad h_3^0 = 0 \]

\[ \mathcal{h} = (h_k^0) = \mathbb{P}_k \]

In general,

\[ \mathcal{h}_k^A = \sum_{e \in S} \mathcal{p}_{ke} \cdot h_e^A \quad \Rightarrow \quad h^A \text{ is a right eigenvector for } \mathcal{P} \text{ w eigenvalue } 1 \]

\[ h_k^A = 1 \text{ if } k \in \mathcal{A} \]

\[ = \text{right e.v.?} \]

There are many!

say \( A \in \mathcal{C} \)
Relation to unfair coin problem:
The vector of hitting probabilities is the minimal non-negative solution to $h = \Phi h$ such that $h_k = 1$ for $k \in A$.

**Proof**. That it is a right-e.v. is clear. What about minimality? Consider approximations $h^{A, m} = \mathbb{P}(X, \text{hits } A \text{ before } m)$.

Obviously, $h^{A, 0} \leq h^{A, 1} \leq \ldots \leq h^{A, m} \leq h^{A, m+1} \leq \ldots \to h^A$. $h^{A, m} = \mathbb{P}^m h^{A, 0}$.

And, $h^A = \lim_{m \to \infty} h^{A, m}$. And (very general!): $h \geq g \Rightarrow \Phi h \geq \Phi g$.

Hence, if $g = \Phi g$, $g \geq h^{A, 0} \Rightarrow g \geq h^{A, m} \Rightarrow g \geq h^A$. \(\square\)

$h_k^{A, m} \leq h_k^{A, m+1}$ for all $k \in S$. $h_k^{A, m} \to h_k^A$.

$h = \Phi h$; \(\square\)
Example

Then $(h_k)$ is characterized by $\frac{p}{k+1} h_{k+1} + q h_{k-1} = h_k \quad [k \geq 1]$

$h_0 = 1$. How to solve recurrence $ah(k) + bh(k+1) + ch(k+2) = 0$?

1. Space of solutions is 2-dimensional, spanned by $\lambda_1^k, \lambda_2^k$, $\lambda_1, \lambda_2$ roots of $a + bh + ch^2 = 0$

   (or, if $\lambda_1 = \lambda_2$, by $\frac{A^k}{k}, \frac{A^k}{k^k}$)

2. In our case, $A_1 = 1, A_2 = \frac{9}{p}$, if $p \neq 9, h = a + b(9/p)^k$

   - If $9/p > 1, 0 < h < 1 \Rightarrow B > 0, h(0) = 1 \Rightarrow A = 1, h(k) = 1$

   - If $9/p < 1, h(k)(9/p)^k + A(1 - (9/p)^k) \geq 0$, minimal solution: $A = 0,$

If $p = 9, h = a + b k, B = 0; A = 1$

I.e. for $p \neq 9$, ruin (absorption at $k = 0$) is prob. 1.

Otherwise, positive prob. to escape...
Example Birth-death processes

\[ 0 \xleftarrow{q_1} 0 \xrightarrow{q_2} 0 \xleftarrow{q_3} 0 \xrightarrow{\ldots} 0 \xleftarrow{q_k} 0 \xrightarrow{\ldots} \]
\[ P_0 \xrightarrow{p_1} P_1 \xrightarrow{p_2} \ldots \xrightarrow{\ldots} P_{k-1} \xrightarrow{p_k} P_k \]

\[ h_0^0 = 1, \quad h_k^0 = q_k h_{k-1}^0 + p_k h_{k+1}^0. \quad \text{We know that} \]
\[ f_k = 1 \quad \text{is a solution} \quad \text{— need something else.} \]

So, consider differences: \( u_k = h_k^0 - h_k^0 \); then

\[ 0 = q_k u_k - p_k u_{k+1} \Rightarrow u_{k+1} = \frac{q_k}{p_k} u_k \]

\[ \Rightarrow u_{k+1} = \frac{q_k q_{k-1} \ldots q_1}{p_k p_{k-1} \ldots p_1} u_1 = \chi_k u_1, \]

So,

\[ h_k = h_0 - u_1 \cdot \sum_{k=0}^{k-1} \frac{q_k q_{k-1} \ldots q_1}{p_k p_{k-1} \ldots p_1}, \quad k \geq 1. \quad (h_0 = 1) \]

1 parameter left, \( u_1 \). Case 1: \( \sum_{k=0}^{\infty} q_k = \infty \). Then \( u_1 = 0, \quad h_k = 1 \).

Case 2: \( \sum_{k=0}^{\infty} q_k = q < \infty \); best \( u_1 = \frac{q}{h_0} \).
So, ultimately

\[
\text{if } \sum g_e = \infty \\
\text{then } h_k^0 = \left( 1 - \frac{k}{\infty} \sum g_e \right) \\
\text{if } \sum g_e = g < \infty, \\
\text{then } h_k^0 = \frac{\sum_{e=k}^{\infty} g_e}{\sum_{e=0}^{\infty} g_e}.
\]

So, if \( \frac{p_k}{q_k} \) is asymptotically not too much below 1, (or, better, is above 1) survival has some chance!

**Example:** Drunkard

\[
h_0^0 = 1; \quad h_N^0 = 0
\]

\[
h_k^0 = \frac{1}{2}(h_{k+1}^0 + h_{k-1}^0).
\]

\[
\Delta^2 h = 0
\]

\[
\Rightarrow h \text{ is a linear function of } k.
\]

\[
\Rightarrow h_k = 1 - \frac{k}{2}
\]
Mean time till absorption \( \overline{w}_{k}^{A} = \overline{E}_{k}(H^{A}) \). \( A \)-closed class

Some trick (almost):

\[
\begin{align*}
\overline{w}_{k}^{A} &= 0 \quad \text{if } k \in A; \\
\overline{w}_{k}^{A} &= 1 + \sum_{k \in A} \overline{w}_{k}^{A}
\end{align*}
\]

I.e.

\[
\overline{w}^{A} = \overline{1}_{A} + \overline{P} \overline{w}^{A} \quad (\approx)
\]

(cannot really invert \((I-P)\), unfortunately).

Thus \( \overline{w}^{A} \) is minimal positive solution to \((\approx)\).

**Proof.** As before, we can represent \( \overline{w}^{A} \) by approximation.

Namely

\[
\begin{align*}
\overline{w}_{k}^{A} &= \overline{E}_{k}(H^{A}) = \sum_{n=1}^{N} \overline{P}_{k}(H^{A} \geq n). \\
\text{Take} \quad \overline{w}_{k}^{A} &= \sum_{n=1}^{N} \overline{P}_{k}(H^{A} \geq n)
\end{align*}
\]

Then

\[
\begin{align*}
\overline{w}_{k}^{A,N+1} &= \overline{P}(H^{A} > 1) + \sum_{n=1}^{N+1} \overline{P}_{k}(H^{A} \geq n) = \overline{1}_{A} + \sum_{n=2}^{N+1} \sum_{k \in A} \overline{P}_{k} \overline{P}_{k}(H^{A} \geq n-1) =
\end{align*}
\]

So,

\[
\begin{align*}
\overline{w}_{k}^{A,N+1} &= \overline{1}_{A} + \overline{P} \overline{w}_{k}^{A,N} = \overline{1}_{A} + \sum_{k \in A} \overline{P}_{k} \overline{w}_{k}^{A,N}
\end{align*}
\]

\[
\overline{w} = \overline{1}_{A} + \overline{P} \overline{w}, \quad \overline{w} \geq 0, \quad \Rightarrow \overline{w} \geq \overline{1}_{A} \overline{w}
\]

\[
\overline{w} = \overline{1}_{A} + \overline{P} \overline{w} \geq \overline{1}_{A} + \overline{P} \overline{w}^{A,N+1} = \overline{w}^{A,N+1} \quad \Rightarrow \quad \overline{w} \geq \lim_{N} \overline{w}^{A,N+1} = \overline{w}^{A}
\]

\[
\overline{w}^{A} \text{ is minimal.}
\]
Exercise. Simulate a Bernoulli($\frac{1}{2}$) coin with a fair one (i.e. create a MC with all transition probabilities $= \frac{1}{2}$, and absorption $h_0^A = \frac{1}{5}$, $h_0^B = \frac{4}{5}$).

Find expected time to absorption.

Example. Find $w_{0, W}^k = v_k$ for the downcard's problem.

\[ w_0^k = w_n^k = 0 \quad j \quad w_k = 1 + \frac{1}{2}(w_{k+1} + w_{k-1}). \]

\[ \Delta^2 w = -2 \implies w_k = a + b k - k^2 \]

\[ \implies w_k = k(N-k). \]