Euler Characteristics of Exotic Configuration Spaces

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1 Introduction

Configuration spaces, - that is the spaces of collections of a fixed number of distinct, distinguishable points in a topological space, - arise in various applications. The topology of such configuration spaces is relatively well understood for manifolds, but for the spaces with singularities, the recorded knowledge is still quite limited.

In this note we will generalize several existing results dealing with the Euler characteristics of variously enhanced configuration spaces.

Those enhancements extend in two directions: first, one can relax the constraints that the points are distinct, allowing some collections of them to occupy the same position, along the lines of the *no-*k*-equal* configuration spaces, see e.g. [BW95]. Second, we allow the points in the collections to be colored, with constraints on their collocation depending on their colors (as in, e.g. [McD75]).

For this general setting, we obtain a general formula for the Euler characteristic of such exotic configuration spaces in a compact stratified set.

1.1 Precursors

Let us start with quoting some of the existing results that we generalize.

1.1.1 Conventional configuration spaces

Let $\mathbb{X}$ be (the geometric realization of) a finite simplicial complex

$$\mathbb{X} = \amalg_\alpha \boldsymbol{\sigma}_\alpha,$$

where $\boldsymbol{\sigma}_\alpha$ are the (relatively) open simplices of the triangulation.

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For a finite set $\mathbf{N} = \{1, \ldots, n\}$, denote by $\mathtt{Conf}(\mathbb{X}, \mathbf{N})$ the configuration space of distinct points indexed by $\mathbf{N}$, i.e.

$$\mathtt{Conf}(\mathbb{X}, \mathbf{N}) := \mathbb{X}^{\mathbf{N}} - \cup_{k \neq l} \Delta_{kl},$$

where $\Delta_{kl} = \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{X}^{\mathbf{N}} : x_k = x_l\}$ are the *big* diagonals in $\mathbb{X}^{\mathbf{N}}$.

For this setup, [Gal01] proves the following

Theorem 1.1 (S. Gal)**.** *The* exponential generating function *for the sequence of Euler characteristics of the configuration spaces of* $\mathbb{X}$ *is given by*

$$\sum_{n \geqslant 0} \chi(\mathtt{Conf}(\mathbb{X}, \mathbf{N})) \frac{z^n}{n!} = \prod_{\alpha} \left(1 + (-1)^{d(\alpha)} (1 - \chi(L(\alpha))) z\right)^{(-1)^{d(\alpha)}}, \tag{1}$$

where the product is taken over the simplices of all dimensions in $\mathbb{X}$*, and* $d(\alpha)$ *is the dimension, and* $L(\alpha)$ *is the* link *of the simplex* $\boldsymbol{\sigma}_\alpha$*.*

(Here the link of a simplex $\boldsymbol{\sigma}_\alpha$ is the simplicial complex formed by all simplices containing α in their boundary, or, equivalently, the base of the cone obtained by the intersection of piecewise linearly embedded $\mathbb{X}$ with a small ball in a linear subspace transversally intersecting α at a single point, around that point.)

1.1.2 No-k-equal spaces

Exponential generating function for Euler characteristics appeared also in the following setting. Take as $\mathbb{X}$ the real line, and consider the *no-*k*-equal* configuration spaces,

$$\mathtt{Conf}_k(\mathbb{R}^1, n) = \mathbb{R}^n - \cup_{I = (i_1 < \ldots < i_k)} \Delta_I, \tag{2}$$

where $\Delta_I = \{x_{i_1} = x_{i_2} = \ldots = x_{i_k}\}$, are k-diagonals, and the union is taken over all such diagonals (which have codimension $(k-1)$). In other words, the no-k-equal configuration space is obtained by forbidding all configurations that have k or more points in the tuple coinciding.

In [BL94] these spaces appeared as an interesting testing ground for topological lower bounds of complexity of decision trees, and were investigated intensely since (mostly from the viewpoint of linear subspace arrangements theory).

One of the results of [BL94] can be formulated as follows:

Theorem 1.2 (A.Björner, L.Lovasz)**.** *The exponential generating function for the Euler characteristics of no-*k*-equal spaces* $\mathtt{Conf}_k(\mathbb{R}^1, \mathbf{N})$ *is given by*

$$\sum_{n \geqslant 0} \chi(\mathtt{Conf}_k(\mathbb{R}^1, n)) \frac{z^n}{n!} = \frac{1}{1 - z + z^2/2 + \ldots \pm z^{k-1}/(k-1)!}. \tag{3}$$

1.2 Colored configuration spaces

The main result in this paper generalizes both theorems above. One specific generalization is to work with *colored* configuration spaces.

1.2.1 Colors and Ideals

Fix a finite set $\mathfrak{C}$ of colors of size c.

The vectors in $\mathbb{N}^{\mathfrak{C}}$ will be referred to as *color counts*.

A subset of color counts $\mathcal{I} \in \mathbb{N}^{\mathfrak{C}}$ is an *ideal* if for any $\mathbf{m} = (m_1, \ldots, m_c) \in I$ and $\mathbf{m}' \leqslant \mathbf{m}$ (this notation shorthands for $m_1' \leqslant m_1, \ldots, m_c' \leqslant m_c$), the point $\mathbf{m}'$ also lies in $\mathcal{I}$. We will be assuming that all the basis vectors of $\mathbb{N}^{\mathfrak{C}}$ are in $\mathcal{I}$.

We will be considering finite collections of distinct colored points in $\mathbb{X}$ numbered by elements of $\mathbf{N}$. For a point $x_k, k \in \mathbf{N}$, we will be denoting to its color as $\mathbf{c}(k)$. If $I \subset \mathbf{N}$, we will denote the corresponding color counts of points in I as $\mathbf{c}(I) \in \mathbf{c}(\mathbf{N})$.

1.2.2 Ideals and Their Configuration Spaces

Fix an ideal $\mathcal{I}$ (of *permissible collisions*).

Definition 1.3. *For a collection of points $\mathbf{x} \in \mathbb{X}^{\mathbf{N}}$ with color count $\mathbf{m} = \mathbf{c}(\mathbf{N})$, the configuration space $\mathtt{Conf}_{\mathcal{I}}(\mathbb{X}, \mathbf{m})$ is defined as*

$$\mathtt{Conf}_{\mathcal{I}}(\mathbb{X}, \mathbf{m}) = \mathbb{X}^{\mathbf{N}} - \bigcup_{I : \mathbf{c}(I) \notin \mathcal{I}} \Delta_I. \tag{4}$$

In words, the configuration space $\mathtt{Conf}_{\mathcal{I}}$ prevents any collection of points in the configuration to coincide if their color counts are outside of the ideal $\mathcal{I}$ of permitted collisions.

Example 1.4. *If $c = 1$, and $\mathcal{I} = \{1\}$, we have the usual configuration spaces; if $\mathcal{I} = \{1, 2, \ldots, k-1\}$, we have the no-$k$-equal configuration spaces.*

The bi-colored ($c = 2$) ideal with $\mathcal{I} = \{(m, 0), (0, m), m = 0, 1, \ldots\}$ forbids any points of different colors to collide, but allows that for any number of points of the same color (the "apartheid" ideal).

Definition 1.5. *The* counting function *of the ideal $\mathcal{I}$ is*

$$\Phi_{\mathcal{I}}(\boldsymbol{z}) = \sum_{\mathbf{m} \in \mathcal{I}} \frac{\boldsymbol{z}^{\mathbf{m}}}{\mathbf{m}!}; \tag{5}$$

here $\boldsymbol{z} = (z_1, \ldots, z_c)$; $\boldsymbol{z}^{\mathbf{m}} = \prod_{k=1}^{c} z_k^{m_k}$ and $\mathbf{m}! = \prod(m_k!)$.

Example 1.6. *For no-k-equal configuration space, $\Phi_{\mathcal{I}}(z) = 1 + z + z^2/2 + \ldots + z^{k-1}/(k-1)!$.*

For the "apartheid ideal", $\Phi_{\mathcal{I}}(z_1, z_2) = \exp(z_1) + \exp(z_2) - 1$.

1.3 Main result

To formulate our main result we need a few more definitions.

1.3.1 Constructible Functions

Let $\mathbb{X}$ be a compact subanalytic set in $\mathbb{R}^D$, with a (finite) Whitney stratification

$$\mathbb{X} = \amalg_\alpha \mathbb{X}_\alpha. \tag{6}$$

We will refer to the functions constant on the strata of $\mathbb{X}$ as *constructible* (for a background on constructible functions, see [Sch91]).

1.3.2 Constructible Euler Characteristic

Definition 1.7. *If $\mathbb{X}$ is a subanalytic triangulated set in $\mathbb{R}^D$, we will refer to the alternating sum of the numbers of simplices of each dimension as its* constructible Euler characteristic, *denoted $\chi_c(\mathbb{X})$. Equivalently, χ_c is the Euler characteristic computed as alternated sum of ranks of Borel-Moore cohomologies.*

It is well known that χ_c is independent of the choice of triangulation, is therefore additive on subanalytic sets (i.e. $\chi_c(A) = \chi_c(A - B) + \chi_c(B)$ for subanalytic $B \subset A$), and matches the standard (homotopy-invariant) Euler characteristic on *compact* subanalytic sets.

One can define the integral of a constructible function with compact support with respect to constructible Euler characteristic χ_c as

$$\int f d\chi_c = \sum_s s\chi_c(f^{-1}(s)),$$

where the sum is taken over the (finite) range of f, or, equivalently, as the evaluation of the direct image of the integrand under the mapping of $\mathbb{X}$ to a point.

Definition 1.8. *For a constructible function f, its* dual f° *is defined as*

$$f^\circ(x) = \int_{\mathbb{X}} \mathbf{1}_{B(x,\epsilon)} f d\chi_c,$$

where $\mathbf{1}_{B(x,\epsilon)}$ is the indicator function of the Euclidean ball of radius $\epsilon > 0$ around x in $\mathbb{R}^D$, and the integral is with respect to Euler characteristic (see, e.g. [GZ10]) (the integral stabilizes when ϵ is small enough, and thus is well-defined).

For such subanalytic $\mathbb{X} \subset \mathbb{R}^D$ we set $\mathbf{1} = \mathbf{1}_{\mathbb{X}}$ to be the indicator function of $\mathbb{X}$. This is a constructible function, as is its dual,

$$\mathbf{1}^\circ(x) = \chi(\mathbb{X} \cap B_\epsilon(x)). \tag{7}$$

It is easy to see that

Lemma 1.9. *The function $\mathbf{1}^\circ_{\mathbb{X}}$ is supported by $\mathbb{X}$ and is constant along the strata of $\mathbb{X}$.*

1.3.3 Main Result

Theorem 1.10. *The exponential generating function for the Euler characteristic of $\mathcal{J}$-configuration spaces is given by*

$$\sum_{\mathbf{m}} \chi(\mathtt{Conf}_{\mathcal{J}}(\mathbb{X}, \mathbf{m}))\frac{\boldsymbol{z}^{\mathbf{m}}}{\mathbf{m}!} = \prod_\alpha \Phi_{\mathcal{J}}(\mathbf{1}^\circ(\alpha)\boldsymbol{z})^{(-1)^{d(\alpha)}\chi(\alpha)}, \tag{8}$$

where the product is taken over all strata of $\mathbb{X}$, $\mathbf{1}^\circ(\alpha)$ is the common value $\mathbf{1}^\circ(x)$ for the points $x \in \alpha$ of the stratum α, $d(\alpha)$ is the dimension, and $\chi(\alpha)$ is the Euler characteristic of the stratum $\mathbb{X}_\alpha$.

Equivalently,

Corollary 1.11.

$$\sum_{\mathbf{m}} \chi(\mathtt{Conf}_{\mathcal{J}}(\mathbb{X}, \mathbf{m}))\frac{\boldsymbol{z}^{\mathbf{m}}}{\mathbf{m}!} = \prod_\alpha \Phi_{\mathcal{J}}(\mathbf{1}^\circ(\alpha)\boldsymbol{z})^{\chi_c(\alpha)}, \tag{9}$$

where $\chi_c(\alpha)$ is the constructible *Euler characteristic of the stratum $\mathbb{X}_\alpha$.*

1.4 Plan

We start with a few examples of applying our main theorem. In Section 3 we will present the proof, drawing on the combinatorial results addressed in Section 4 and geometric results presented in Section 5.

2 Examples

2.1 Gal's formula

Let $\mathbb{X}$ be a triangulated space, embedded into some Euclidean space so that the embedding is linear on each simplex. We consider $\mathbb{X}$ as stratified by these simplices. For a point in a simplex of dimension d, the intersection of an open small ball with $\mathbb{X}$ is homeomorphic to $\mathbb{R}^d \times K$, where K is the open cone with the base $L(x)$, the link of $\mathbb{X}$ at x. Hence,

$$\mathbf{1}^\circ(x) = (-1)^d(1-\chi(L(x))). \tag{10}$$

For "conventional" configuration spaces, $\mathcal{J} = \{0, 1\}$, and $\Phi_{\mathcal{J}}(z) = 1 + z$. Substituting these identities into (6) leads to

$$\sum_n \chi(\mathtt{Conf}(\mathbb{X}, n))\frac{z^n}{n!} = \prod_\alpha (1 + (-1)^{d(\alpha)}(1-\chi(L(\alpha))))^{(-1)^{d(\alpha)}}, \tag{11}$$

i.e. Gal's formula.

2.2 No-k-equal configuration spaces

For no-k-equal spaces, the counting function is $\Phi_{\mathcal{J}}(z) = 1 + z + \ldots + z^{k-1}/(k-1)! =: e_k(z)$.

If $\mathbb{X}$ is a d-dimensional disk, $\mathbf{1}^\circ = (-1)^d$ in the interior of the disk, and 0 on the boundary. This implies

$$\sum_n \chi(\mathtt{Conf}_k(\mathbb{R}^d, n))\frac{z^n}{n!} = e_k((-1)^d z)^{(-1)^d}, \tag{12}$$

recovering, in particular, the formula (3).

2.3 Graphs

If $\mathbb{X}$ is a graph, i.e. the geometric realization of a one-dimensional simplicial complex, one has for any $\mathcal{J}$

$$\sum_{\mathbf{n}} \chi(\mathtt{Conf}_{\mathcal{J}}(\mathbb{X}, \mathbf{n}))\frac{\mathbf{z}^{\mathbf{n}}}{\mathbf{n}!} = \frac{\prod_{\alpha\in V(\mathbb{X})} \Phi_{\mathcal{J}}((1-\nu(\alpha))\mathbf{z})}{\Phi_{\mathcal{J}}^{|E(\mathbb{X})|}(-\mathbf{z})}, \tag{13}$$

where the product is taken over the vertices α, and $\nu(\alpha)$ stands for the degree of α; E is the set of the edges in the graph.

We remark, in particular, that the leaf vertices do not contribute, and each degree 2 vertex cancels one edge (as it should, to maintain the invariance of $\mathtt{Conf}_{\mathcal{J}}$ with respect to subdivisions).

2.4 Manifolds

If $\mathbb{X}$ is a compact manifold of dimension d, one has but one stratum, and

$$\sum_{\mathbf{n}} \chi(\mathtt{Conf}_{\mathcal{J}}(\mathbb{X}, \mathbf{n})) \frac{z^{\mathbf{n}}}{\mathbf{n}!} = \Phi_{\mathcal{J}}((-1)^d z)^{(-1)^d \chi(\mathbb{X})}. \tag{14}$$

In particular, for finite ideals $\mathcal{J}$ and even-dimensional manifolds, $\chi(\mathtt{Conf}_{\mathcal{J}}(\mathbb{X}, \mathbf{n})) = 0$ for large enough $\mathbf{n}$ (and is $\equiv 0$ for odd-dimensional manifolds).

2.5 Bicolored spaces

Consider now an example of configuration spaces with infinite ideals: the bi-colored apartheid ideal, with $\Phi_{\mathcal{J}} = \exp(z_1) + \exp(z_2) - 1$. Using (14) we obtain

$$\sum_{n_1, n_2} \chi(\mathtt{Conf}_{\mathcal{J}}(S^2, (n_1, n_2))) \frac{z_1^{n_1} z_2^{n_2}}{n_1! n_2!} = (\exp(z_1) + \exp(z_2) - 1)^2. \tag{15}$$

In particular, all Euler characteristics $\chi(\mathtt{Conf}_{\mathcal{J}}(S^2, (n_1, n_2))) = 2$ if $n_1, n_2 \geqslant 1$.

3 Proof of Theorem 1.11

3.1 Setting the Stage

Fix a finite set $\mathbf{N}$ of points with colors in $\mathfrak{C}$.

3.1.1 Partitions

We will denote the set of partitions of $\mathbf{N}$ as $\mathtt{Part}(\mathbf{N})$. (A partition is an unordered collection of disjoint subsets of $\mathbf{N}$ exhausting it.)

Partitions of $\mathbf{N}$, or, more generally, of any finite set S, are partially ordered by refinement: here we write $\sigma \prec \pi$ if σ is a refinement of π, or π is a coarsening of σ, that is if each block of σ is contained entirely within a block of π.

The minimal (with respect to this refinement order) partition of S consisting of singletons will be denoted as $\mathbf{0}_S$, or just $\mathbf{0}$, when the context is clear.

The *join* of a family of partitions is the minimal common coarsening of the partitions in the family.

The maximal element for the poset $\mathtt{Part}(S)$, denoted as $\mathbf{1}_S$ is the partition with one block, (S).

3.1.2 Diagonals

For each partition $\pi = (\pi_1)(\pi_2)\dots(\pi_k)$ of $\mathbf{N}$ into k blocks, we will form the diagonal

$$\boldsymbol{\Delta}(\pi) = \{x : x_k = x_l \text{ if } k, l \in \text{ same block of } \pi\}. \tag{16}$$

(Clearly, the diagonals form a lattice isomorphic to the lattice of partitions of $\mathbf{N}$ with reversed order.)

Fix colors of each of the points in $\mathbf{N}$; set $\mathbf{n} = \mathbf{c}(\mathbf{N}) = (n_1, \ldots, n_c)$ to be the vector of color counts of $\mathbf{N}$.

Definition 3.1. *We say that a partition is* forbidden*, if for at least one block, the color count of this block is not in* $\mathcal{J}$.

We denote the collection of forbidden partitions as $\mathtt{Part}^{\mathtt{F}}_{\mathcal{J}}(\mathbf{N})$.

Then, manifestly,

$$\mathtt{Conf}_{\mathcal{J}}(\mathbb{X}, \mathbf{n}) = \mathbb{X}^{\mathbf{N}} - \boldsymbol{\Delta}_{\mathtt{Part}^{\mathtt{F}}_{\mathcal{J}}}, \text{ where } \quad \boldsymbol{\Delta}_{\mathtt{Part}^{\mathtt{F}}_{\mathcal{J}}} = \bigcup_{\pi \in \mathtt{Part}^{\mathtt{F}}_{\mathcal{J}}(\mathbf{N})} \boldsymbol{\Delta}(\pi). \tag{17}$$

3.1.3 Enabling Additivity

One certainly can find the (constructible) Euler characteristic of $\mathtt{Conf}_{\mathcal{J}}$ using (17) via additivity and some form of inclusion-exclusion principle. However this constructible $\chi_c(\mathtt{Conf}_{\mathcal{J}})$ is not equal to its homotopy invariant χ as the configuration space is not compact.

To circumvent this problem, we replace the diagonals in (17) by their appropriately chosen open vicinities ("fattenings") in $\mathbb{X}^{\mathbf{N}}$. If the compact constructible complement to those vicinities is homotopy equivalent to the corresponding configuration space, we can use additivity of χ_c to compute $\chi(\mathtt{Conf}_{\mathcal{J}})$.

3.1.4 Combinatorics

The combinatorial component of the computation is rather standard, and amounts to the manipulations with Möbius functions for the poset of partitions of colored sets of points. These computations allow one to reduce the problem to finding the constructible Euler characteristics of fattenings of diagonals. The proofs of there combinatorial results are given in Section 4.

3.1.5 Geometry and the Rest

The definitions of the fattenings of the diagonals and evaluations of their Euler characteristics is done in Section 5.

Combining all the elements together leads to some formal power series manipulations which is shown below.

3.2 Proof of the Main Theorem

We start with a straightening of the geometry of $\mathbb{X}$.

3.2.1 Cubical embedding

Namely, it will be convenient to *cubulate* the space $\mathbb{X}$. Let $\mathbb{I} = [0, 1]$ be the unit interval.

Proposition 3.2. *There exists a refinement of the stratification of $\mathbb{X}$ which is homeomorphic to a subcomplex of the natural cubical complex of the unit cube $\mathbb{I}^D$ in some Euclidean space.*

The proof is given in Section 5. □

From now on, we will assume that $\mathbb{X}$ is a cubical subcomplex of I^D, equipped with the metric induced from the $\sup$ norm on $\mathbb{R}^D$.

3.2.2 Deformation Retract

For $\mathbf{x} \in \mathbb{X}^{\mathbf{N}}$, we define the partition $\pi(\mathbf{x}, \epsilon)$ as the partition of $\mathbf{N}$ into the classes of equivalence defined by the transitive closure of the relation

$$x_k \sim x_l \Leftrightarrow |x_k - x_l| < \epsilon.$$

Definition 3.3. *We define (ϵ-)fattening of $\mathbf{\Delta}^\epsilon(\pi)$ as the set of configurations $\mathbf{x} \in \mathbb{X}^{\mathbf{N}} : \pi \preceq \pi(\mathbf{x}, \epsilon)$, and*

$$\mathbf{\Delta}^\epsilon(\mathcal{J}) := \bigcup_{\pi \in \mathtt{Part}^F_{\mathcal{J}}(\mathbf{N})} \mathbf{\Delta}^\epsilon(\pi).$$

Clearly, $\mathbf{\Delta}^\epsilon(\mathcal{J})$ is an open vicinity of $\mathbf{\Delta}_{\mathtt{Part}^F_{\mathcal{J}}}$, and

$$\mathtt{Conf}^\epsilon_{\mathcal{J}}(\mathbb{X}, \mathbf{N}) = \mathbb{X}^{\mathbf{N}} - \mathbf{\Delta}^\epsilon(\mathcal{J}) \tag{18}$$

is a compact subset of $\mathtt{Conf}_{\mathcal{J}}(\mathbb{X}, \mathbf{N})$.

Proposition 3.4. *Under the assumptions of 3.2.1, $\mathtt{Conf}^\epsilon_{\mathcal{J}}(\mathbb{X}, \mathbf{N})$ is a deformation retract of $\mathtt{Conf}_{\mathcal{J}}(\mathbb{X}, \mathbf{N})$ for small enough ϵ.*

The proof is given in Section 5. □

3.2.3 Inclusion-Exclusion Formulae

Proposition 3.5. *There exists a function $c_{\mathcal{J}}$ of partitions of $\mathbf{N}$ such that*

$$\sum_{\sigma \preccurlyeq \pi} c_{\mathcal{J}}(\sigma) = \begin{cases} 0 & \text{if } \pi \in \mathtt{Part}^F_{\mathcal{J}}(\mathbf{N}), \\ 1 & \text{otherwise} \end{cases} \tag{19}$$

For single blocks, this function depends only on the color content $c_{\mathcal{J}}(\mathbf{N}) = c_{\mathcal{J}}(\mathbf{c}(\mathbf{N}))$, and is multiplicative, in the sense that

$$c_{\mathcal{J}}((\pi_1) \dots (\pi_k)) = \prod_l c_{\mathcal{J}}(\mathbf{c}(\pi_l)). \tag{20}$$

The proof is in Section 4. □

Lemma 3.6. *For sufficiently small* ϵ*, the intersection lattice generated by the sets* $\mathbf{\Delta}^{\epsilon}(\pi), \pi \in \mathtt{Part}(\mathbf{N})$ *is isomorphic to the partition lattice on* $\mathbf{N}$*: for any family of partitions* $\{\pi_\lambda\}, \lambda = 1, \dots, \Lambda$*, the fattening of their join* $\pi = \pi_1 \wedge \dots \wedge \pi_\Lambda$ *equals the intersection of the fattenings of the partitions in the family:*

$$\mathbf{\Delta}^{\epsilon}(\pi) = \bigcap \mathbf{\Delta}_{\epsilon}(\pi_k). \tag{21}$$

Proof. This follows directly from the definition of fattenings $\mathbf{\Delta}^{\epsilon}(\pi)$. □

We remark that this implies that $\mathbf{\Delta}^{\epsilon}(\pi) \subset \mathbf{\Delta}^{\epsilon}(\pi')$ for any refinement π' or π.

The Proposition 3.5 and Lemma 3.6 allow us to represent the (topological) Euler characteristics in terms of the (constructible) Euler characteristics of the diagonal fattenings.

Proposition 3.7.

$$\chi(\mathtt{Conf}_{\mathcal{J}}(\mathbb{X}, \mathbf{n})) = \sum_{\pi \in \mathtt{Part}(\mathbf{n})} c_{\mathcal{J}}(\pi) \chi_c(\mathbf{\Delta}^{\epsilon}(\pi)). \tag{22}$$

Proof. For $x \in \mathbb{X}^{\mathbf{N}}$ denote by $\pi(x)$ the coarsest partition π for which $x \in N^{\epsilon}(\pi)$ (it exists by Lemma 3.6). Then

$$\sum_{\sigma \in \mathtt{Part}(\mathbf{n})} c_{\mathcal{J}}(\sigma) \mathbf{1}_{\mathbf{\Delta}^{\epsilon}(\sigma)}(x) = \sum_{\sigma \preceq \pi(x)} c_{\mathcal{J}}(\sigma) \tag{23}$$

which is 1 exactly when $x \in \mathbb{X}^{\mathbf{N}} - \bigcup_{\pi \in \mathtt{Part}^{F}_{\mathcal{J}}(\mathbf{N})} \mathbf{\Delta}^{\epsilon}(\pi)$. Invoking the additivity of the constructible Euler characteristics completes the proof. □

3.2.4 Computations

We will need also the multiplicativity of the Euler characteristics of $\mathbf{\Delta}^{\epsilon}(\pi)$:

Lemma 3.8. *If* $(\pi_1) \dots (\pi_p)$ *are the blocks of the partition* π*, then*

$$\chi_c(\mathbf{\Delta}^{\epsilon}(\pi)) = \prod_l \chi_c(\mathbf{\Delta}^{\epsilon}(\pi_l)).$$

(*Here the notation* $\mathbf{\Delta}^{\epsilon}(\pi_l)$ *is used to denote the fattening of the main diagonal in* $\mathbb{X}^{\pi_l}$ *- remark that it depends only on the size of the block* π_l*.*)

Proof. The fact that a subset $I = (k_1, \dots, k_p)$ of $\mathbf{N}$ belongs to a part of $\pi(\boldsymbol{x}, \epsilon)$ depends only on the points $x_{k_1}, \dots, x_{k_p}$ implies that the set $\mathbf{\Delta}_{\epsilon}(\pi)$ is a product of its projections to $\mathbb{X}^{\pi_k}$, over all parts of $\pi = (\pi_1) \cdots (\pi_p)$. □

Using Proposition 3.7 we can transform the exponential generating function as

$$F(\boldsymbol{z}) = \sum_{\mathbf{n}} \frac{\boldsymbol{z}^{\mathbf{n}}}{\mathbf{n}!} \chi(\mathtt{Conf}_{\mathcal{J}}(\mathbb{X}, \mathbf{n})) = \sum_{\mathbf{n}} \frac{\boldsymbol{z}^{\mathbf{n}}}{\mathbf{n}!} \sum_{\substack{\pi \in \mathtt{Part}(\mathbf{n}) \\ \pi = (\pi_1) \dots (\pi_k)}} \prod_{l=1}^{k} c_{\mathcal{J}}(\mathbf{n}(\pi_l)) \chi_c(\mathbf{\Delta}^{\epsilon}(\pi_l)). \tag{24}$$

Lemma 3.9.

$$F(\boldsymbol{z}) = \exp\left(\sum_{\mathbf{m} > \mathbf{0}} \frac{\boldsymbol{z}^{\mathbf{m}}}{\mathbf{m}!} c_{\mathcal{J}}(\mathbf{m}) \chi_c(\mathbf{\Delta}^{\epsilon}(|\mathbf{m}|)) \right). \tag{25}$$

Proof. Denoting

$$\kappa(\mathbf{m}) := c_{\mathcal{J}}(\mathbf{m})\chi_c(\boldsymbol{\Delta}^\epsilon(|\mathbf{m}|)),$$

and using Lemma 4.1 as

$$\exp\left(\sum_{\mathbf{m}>\mathbf{0}} \frac{\boldsymbol{z}^{\mathbf{m}}}{\mathbf{m}!}\kappa(\mathbf{m})\right) = \sum_{\mathbf{n}} \frac{\boldsymbol{z}^{\mathbf{n}}}{\mathbf{n}!} \sum_{\substack{\pi\in\mathtt{Part}(\mathbf{n}) \\ \pi=(\pi_1)\dots(\pi_k)}} \prod_{l=1}^{k} \kappa(\mathbf{n}(\pi_l))$$

we arrive at the claim. □

3.2.5 Euler Characteristic of Fattenings

In view of Lemma 3.8, let us turn to $\chi(\boldsymbol{\Delta}^\epsilon(S))$, where $S \subset \mathbf{N}$ has size $|S| =: s$.

Proposition 3.10. *The constructible Euler characteristic of $\boldsymbol{\Delta}^\epsilon(S)$ is*

$$\chi_c(\boldsymbol{\Delta}^\epsilon(S)) = \sum_{\text{faces } \sigma \text{ of } \mathbb{X}} (-1)^{\dim}(\sigma)\chi_c(B_\sigma)^s = \sum_{\text{faces } \sigma \text{ of } \mathbb{X}} \chi_c(\sigma)\chi_c(B_\sigma)^s,$$

where the sum is taken over all open cubes σ of the cubulation of $\mathbb{X}$, and B_σ is the intersection of small enough ball centered at a point of σ with $\mathbb{X}$.

Corollary 3.11. *The Euler characteristic of $\boldsymbol{\Delta}^\epsilon(S)$ is*

$$\chi(\boldsymbol{\Delta}^\epsilon(S)) = \sum_{\text{strata } \mathbb{X}_\beta \text{ of } \mathbb{X}} (-1)^{\dim}(\mathbb{X}_\beta)\mathbf{1}^\circ(\mathbb{X}_\beta)^s, \tag{26}$$

where the sum is taken over all strata of $\mathbb{X}$.

Proof. Follows from the previous Proposition and the immediate fact that along a stratum, the constructible Euler characteristic of the intersection of a small ball with the subanalytic space $\mathbb{X}$ does not depend on the simplex (cube) of a triangulation (cubulation) compatible with the stratification. □

3.2.6 Final Strokes

Substituting (26) into (25) we obtain

$$\sum_{\mathbf{m}} \frac{\boldsymbol{z}^{\mathbf{m}}}{\mathbf{m}!}\kappa(\mathbf{m}) = \sum_{\mathbf{m}} c_{\mathcal{J}}(\mathbf{m})\frac{\boldsymbol{z}^{\mathbf{m}}}{\mathbf{m}!}\left(\sum_{\text{strata } \mathbb{X}_\beta} (-1)^{\dim}(\mathbb{X}_\beta)\mathbf{1}^\circ(\mathbb{X}_\beta)^{|\mathbf{m}|}\right), \tag{27}$$

which after swapping the order of summations becomes

$$\sum_{\text{strata } \mathbb{X}_\beta} (-1)^{\dim}(\mathbb{X}_\beta)\left(\sum_{\mathbf{m}} c_{\mathcal{J}}(\mathbf{m})\frac{(\mathbf{1}^\circ(\mathbb{X}_\beta)\boldsymbol{z})^{\mathbf{m}}}{\mathbf{m}!}\right). \tag{28}$$

Proposition 3.12. *Under the assumptions on the ideal $\mathcal{I}$,*

$$\sum_{\mathbf{m}>\mathbf{0}} c_{\mathcal{I}}(\mathbf{m}) \frac{\boldsymbol{z}^{\mathbf{m}}}{\mathbf{m}!} = \log\left(1 + \sum_{\mathbf{n}\in\mathcal{I}} \frac{\boldsymbol{z}^{\mathbf{n}}}{\mathbf{n}!}\right). \tag{29}$$

The proof is given in Section 4. □

Finishing the proof of the Main Theorem: Plugging (29) into (25) we obtain the Theorem 1.11. □

4 Combinatorics

4.1 Posets and Möbius functions

4.1.1 Partitions and Exponents

The following Lemma is standard:

Lemma 4.1. *Let $\kappa : \mathbb{N}^{\mathfrak{C}} \to \mathbb{C}$ be an arbitrary valuation such that $\kappa(\mathbf{0}) = 0$. Then*

$$\exp\left(\sum_{\mathbf{m}} \frac{\boldsymbol{z}^{\mathbf{m}}}{\mathbf{m}!}\kappa(\mathbf{m})\right) = \sum_{\mathbf{n}} \frac{\boldsymbol{z}^{\mathbf{n}}}{\mathbf{n}!} \sum_{\substack{\pi\in\mathtt{Part}(\mathbf{n}) \\ \pi=(\pi_1)\dots(\pi_k)}} \prod_{l=1}^{k} \kappa(\mathbf{n}(\pi_l)).$$

Proof. See discussion of partitional composites of structures in [BBLL98]. □

4.1.2 Möbius Inversion

Proof of Proposition 3.5. The existence of the function $c_{\mathcal{I}}$ satisfying the conditions (19) is immediate, as the conditions imposed on it are "upper-triangular". More explicitly, one can express $c_{\mathcal{I}}$ in terms of the Möbius function of the poset of partitions:

$$c_{\mathcal{I}}(\pi) = \sum_{\sigma\preceq\pi} \mu(\sigma,\pi) d_{\mathcal{I}}(\sigma), \tag{30}$$

where μ is the Möbius function of the poset `Part`, and $d_{\mathcal{I}}$ is the right hand side of (19), i.e. the indicator function that each block of π has color counts in $\mathcal{I}$.

For $\pi = (\pi_1)\cdots(\pi_p)$, the poset of partitions refining π is a direct product of the partitions of the blocks π_k of π, and so the Möbius function factors. Also, as follows from the definition, the indicator function that the partition π is admissible is the product of indicator functions that each block is admissible. Therefore, $c_{\mathcal{I}}$ itself factors. □

Last combinatorial proof:

Proof of Proposition 3.12: This, again, follows from Lemma 4.1: indeed, it implies that

$$\exp\left(\sum_{\mathbf{m}>\mathbf{0}} c_{\mathcal{J}}(\mathbf{m})\frac{\mathbf{z}^{\mathbf{m}}}{\mathbf{m}!}\right) = \sum_{\mathbf{n}} \frac{\mathbf{z}^{\mathbf{n}}}{\mathbf{n}!} \sum_{\pi\in\mathtt{Part}(\mathbf{n})} c_{\mathcal{J}}(\pi), \tag{31}$$

the exponential generating function of sums

$$\sum_{\pi\in\mathtt{Part}(\mathbf{n})} c_{\mathcal{J}}(\pi)$$

over all partitions π of a set of color counts $\mathbf{n}$. This, by definition is 1 exactly when $\mathbf{n} \in \mathcal{J}$ or $\mathbf{n} = \mathbf{0}$. □

5 Geometry

5.1 Simplifying the Space

5.1.1 Cubulations

Proof of Proposition 3.2: Fix a finite triangulation of $\mathbb{X}$ (existing for any subanalytic set, see, e.g. [Loj64]), and realize $\mathbb{X}$ as a subcomplex of the unit simplex $\Sigma = \{\sum_j x_j = 1, x_j \geqslant 0, j = 1, \ldots, D\} \subset \mathbb{R}^D$ in D-dimensional Euclidean space. Consider the mapping $s : \Sigma \to \mathbb{I}^D$ given by

$$s : \mathbf{y} = (y_1, \ldots, y_D) \mapsto (\max_k y_k)^{-1}\mathbf{y}. \tag{32}$$

It is bijective onto its image, takes each simplicial face of Σ into a union of faces of $\mathbb{I}^D$, and respects adjacencies. □

To prove the deformation retraction of Proposition 3.4 we will use a dynamical system on the n-dimensional cube $\mathbb{I}^{\mathbf{N}}$.

Lemma 5.1. *There exists a smooth vector field $v(\mathbf{z})$ on $\mathbb{I}^{\mathbf{N}} = \{(z_1, \ldots, z_{\mathbf{N}}), 0 \leqslant k \leqslant 1, k = 1, \ldots, \mathbf{N}\}$ such that*

- *the order of coordinates $\{z_k\}$ on the interval is preserved by the flow;*
- *consequently, the diagonals $\{z_k = z_l\}$ are preserved by the flow;*
- *the facets $\{z_k = 0\}$ and $\{z_k = 1\}$ of $\mathbb{I}^{\mathbf{N}}$ are preserved by the flow;*
- *the distances between the coordinates are increasing, unless they are at least $1/(n+1)$;*

Proof. There are many options. Perhaps the easiest is the vector field defined on the simplices into which $\mathbb{I}^{\mathbf{N}}$ is partitioned by the big diagonals $z_j = z_k$, i.e. the simplices consisting of vectors whose coordinates $z_k, k = 1, \ldots, n$ are ordered in a particular way. Such a simplex can be coordinatized by the lengths of consecutive intervals $y_k := z_{(k+1)} - z_{(k)}$ between the reordered coordinates $z_{(1)} \leqslant \ldots \leqslant z_{(n)}$, where $\{z_{(1)}, \ldots, z_{(n)}\} = \{z_1, \ldots, z_n\}$ is a non-decreasing reordering of the coordinates (we also add $z_{(0)} := 0; z_{(n+1)} := 1$).

To define the vector field on each of such n-dimensional simplices, we set

$$\dot{y}_k = y_k \left(\sum_0^n y_l^2 - y_k \right). \tag{33}$$

One can verify immediately that the flow preserves the simplex $\{y_k \geqslant 0; \sum_k y_k = 1\}$, thus defining a vector field on the cube $\mathbb{I}^{\mathbf{N}}$ and that all the properties above are satisfied. $\square$

Proof of Proposition 3.4. It is enough to construct a flow on $(\mathbb{I}^D)^{\mathbf{N}}$ which preserves $\mathbb{X}^{\mathbf{N}}$, and such that any trajectory starting at a point of the configuration space $\mathtt{Conf}_{\mathcal{J}}(\mathbb{X}, \mathbf{N})$ reaches the complement of $\boldsymbol{\Delta}^\epsilon(\mathcal{J})$.

We represent $(\mathbb{I}^D)^{\mathbf{N}}$ as $(\mathbb{I}^{\mathbf{N}})^D$ and apply the flows ν separately to each of the coordinate cubes $\mathbb{I}^{\mathbf{N}}$, so that the evolutions of n-tuples of j-th coordinates of points in $\mathbf{N}$ are independent and given by the vector field ν constructed in Lemma 5.1.

The trajectories of this flow will increase the nonzero gaps between the coordinates until they reach at least $1/(n+1)$.

If $\epsilon < 1/(n+1)$, then the trajectories of a point outside a particular diagonal $\boldsymbol{\Delta}(\pi)$ will be eventually leave the vicinity $\boldsymbol{\Delta}^\epsilon(\pi)$, with the exit point depending continuously on the initial point. This, clearly, proves the claim. $\square$

5.2 Euler Characteristics of the Vicinities of Diagonals

We will need a few intermediate constructions.

For $x \in \mathbb{X}$, we denote by $\sigma(x)$ the face of $\mathbb{X}$ (which, again, we consider as a subcomplex of $\mathbb{I}^D$) whose interior contains x.

5.2.1 Vicinities of Faces of $\mathbb{I}^D$

By Lemma 3.8, to compute $\chi_c(\boldsymbol{\Delta}^\epsilon(\pi))$, it is enough to prove Proposition 3.10 computing the (constructible) Euler characteristics of the ϵ-fattening of main diagonal in $\mathbb{X}^S$ for finite subsets of $\mathbf{N}$.

For a face σ of the cube $\mathbb{I}^D$, we will denote by σ_ϵ the $\sqrt{\epsilon}$-vicinity (in $\sup$-norm) of that face in the cube. (We choose this size to go to 0 with ϵ, but to be asymptotically much larger than the size of the fattening of the diagonal we deal with.)

It is immediate that for $\epsilon < 1/2$,

$$\bigcap_k (\sigma_k)_\epsilon = \left(\bigcap_k \sigma_k \right)_\epsilon. \tag{34}$$

We denote by

$$\bar{\sigma}_\epsilon := \sigma_\epsilon - \cup_{\sigma' \prec \sigma} \sigma'_\epsilon$$

the rectified vicinity of the face σ; clearly $\bar{\sigma}_\epsilon$ is a product of (closed) $\sqrt{\epsilon}$-offset of σ times the $\sqrt{\epsilon}$-vicinity in the normal to σ subspace. Obviously, the constructible sets $\bar{\sigma}_\epsilon$ form a partition of $\mathbb{I}^D$.

5.2.2 Centers of Mass and their Snappoints

Let $S \in \mathbf{N}$ be a nonempty subset. For a tuple of points $\mathbf{x} = (x_1, \dots, x_S) \in \mathbb{X}^S \subset (\mathbb{I}^D)^S$, define

$$m(\mathbf{x}) = \frac{1}{S}\sum_{k=1}^{S} x_k \in \mathbb{I}^D$$

to be their their center of gravity.

We will call the (unique) face σ such that $m(\mathbf{x}) \in \bar{\sigma}_\epsilon$ the *snap point* of $\mathbf{x}$, and denote it as $\sigma(\mathbf{x})$.

Lemma 5.2. *For ϵ small enough, if $\sigma(\mathbf{x})$ is the snap point of $\mathbf{x} = (x_1, \dots, x_s) \in \boldsymbol{\Delta}^\epsilon(S)$, then all faces $\sigma_k := \sigma(x_k)$ are adjacent to $\sigma(\mathbf{x}) : \sigma(\mathbf{x}) \preceq \sigma_k, k = 1, \dots, s$.*

Proof. This follows immediately from (34). □

In particular, the snap point of $\mathbf{x} \in \boldsymbol{\Delta}^\epsilon(S)$ is in $\mathbb{X}$, and we obtain partition of $\boldsymbol{\Delta}^\epsilon(S)$ by the constructible sets

$$\boldsymbol{\Delta}^\epsilon(S) = \amalg_{\sigma \in \mathbb{X}} \sigma^\epsilon, \tag{35}$$

where

$$\sigma^\epsilon = \{\mathbf{x} : \sigma(\mathbf{x}) = \sigma\}. \tag{36}$$

Now we are ready to finish

the Proof of Proposition 3.10. For $\mathbf{x} \in \boldsymbol{\Delta}^\epsilon(S)$, we denote by $\mu(\mathbf{x})$ the orthogonal projection of $m(\mathbf{x})$ to the snap point σ of $\mathbf{x}$. Clearly, $\mu(\mathbf{x}) \in \bar{\sigma}_\epsilon$. Thus the constructible set σ^ϵ is fibered over $bSe \cup \sigma$. The fiber over a point $x \in \bar{\sigma}_\epsilon$,

$$K_x^\epsilon := \{\mathbf{x} \in \boldsymbol{\Delta}^\epsilon : \mu(\mathbf{x}) = x\} \tag{37}$$

is a constructible set. The tangent cone to K_x^ϵ splits into the subspace of $(T_x\sigma)^S$ consisting of tangent vectors adding up to 0, and S-th Cartesian power of the tangent cone L_σ to the intersection of normal space to σ with $\mathbb{X}$. (Clearly, this tangent cone depends only on the face σ, not on a specific point in its interior.)

Each ray along a vector in that tangent cone intersects K_x^ϵ over a half-open interval, and therefore

$$\chi_c(K_x^\epsilon) = (-1)^{\mathrm{dimfc}(s-1)}\chi_c(L_\sigma)^s = (-1)^{\dim(\sigma)}\chi_c(B_\sigma)^s, \tag{38}$$

where B_σ is the constructible Euler characteristic of the intersection of a small ball centered a point in the interior of σ with $\mathbb{X}$.

Now it remains to integrate the Euler characteristics of K_s^ϵ over all possible $x \in \bar{\sigma}_\epsilon$ and apply the Fubini theorem. This proves the claim. □

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