1 Introduction

Quantum random walks (QRWs) is a class of unitary evolution operators that combine the geometry of the underlying space, indexing the positions of the walk, and the internal state dynamics. The literature on quantum random walks is vast, and still is growing, - for more recent survey see, e.g. [VA12].

We deal here with the so-called coined quantum random walks in discrete time on a lattice. The coin remains the same, for all time and space positions, and this translation invariance of the walks we consider allows one to resort to one or another version of Fourier transform, and to understand the corresponding evolution quite precisely.

In particular, one knows that the behavior of such QRWs is ballistic, that is the amplitudes are spreading on a linear (in time) scale over the lattice. These amplitudes exhibit some rather intricate dependence on the data defining the walks, addressed in quite a few papers.

One of the features one can observe experimentally, for the 1- and 2-dimensional lattices, is that the amplitudes, and the corresponding probabilities, form fascinating moire-like patterns, depending on the coin and the jump map (see the definitions below). One of the motivation behind this research was to understand, what is the typical behavior of such patterns.

More specifically, the question we sought to answer is: when the amplitudes are averaged over the random coin and initial internal state, what are the resulting probabilities to find the the particle in a given position?

This randomness is different from often considered random coin model, where the realization of the coin differ frm time to time, or from site to site. (In this case, the behavior is not ballistic, but rather diffusive, see e.g. [AVWW11].) In our situation, for a given coin we observe some ballistic propagation pattern, and then average these patterns over the coins.

The natural probability measure on the unitary coins is the Haar measure over the unitary group SU(c). It turns out that in this case the question can be answered precisely (see Theorem 5.1): the averaged probability is the push-forward under the jump map of the uniform measure on the simplex spanned by the internal states. The intricate patterns boringly add up to a spline.

Besides that result, a few other novel (I believe) results are presented in this note. Thus, in the section 3 we sketch a new proof of the characterization of the weak limits of the (scaled) position of the QWR in terms of the Gauss map. Further, in section 4 we address the localization of QRWs, showing that the strong localization is equivalent to the weak one, and prove that the localization speeds are quite constrained.

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1.1 Translation Invariant Lattice QRWs

To define a (translation invariant) quantum random walk on a lattice one needs the following data:

- a lattice \( L \subset \mathbb{R}^d \) of rank \( d \). In what follow, we will be just assuming \( L = \mathbb{Z}^d \).
- a chirality space, that is a \( c \)-dimensional Hilbert space \( H \cong \mathbb{C}^c \) with a fixed orthonormal basis \( C := \{e_1, \ldots, e_c\} \);
- a coin: a unitary operator \( U \) on the chirality space \( H \);
- the jump map: a mapping \( j : C \to L \). We will assume (without loss of generality) that the jumps \( j_k = j(e_k), k = 1, \ldots, c \) span \( \mathbb{R}^d \) affinely.

Tensoring \( l_2(L) \) with \( H \) results in the Hilbert space \( H_L := l_2(L) \otimes H \) with the basis \( |k, v\rangle, k \in L, v \in \mathbb{C}^c \). Notation: we will be using \( \langle \cdot, \cdot \rangle \) for the (standard) Euclidean product in \( \mathbb{R}^d \), the ambient space of the lattice, and \( \langle \cdot, \cdot \rangle \) for the Hermitian product in \( H \) or \( H_L \).

1.1.1 Defining Quantum Random Walk

The quantum random walk associated with these data is then the unitary evolution on \( H_L \) defined as the composition of two operators, \( S = S_2 \circ S_1 \), which are defined, in turn, as follows:

The operator \( S_1 \) applies the coin at each site of the lattice, that is

\[
S_1 = 1d_{l_2(L)} \otimes U.
\]

In other words, \( U \) acts on each \( H_k, k \in L \) independently.

Second operator step consists of shifting each of the “layers” \( l_2(L) \otimes e_k, k = 1, \ldots, c \) by \( j(e_k) \)

\[
S_2 : |k \otimes e_k\rangle \mapsto |(k + j(e_k)) \otimes e_k\).
\]

1.1.2 Evolution of Coined QRWs

The evolution defined by \( \{S^T\}_{T \in \mathbb{N}} \) exhibits ballistic behavior, that is the effective support of the amplitudes in \( L \) grows linearly with \( T \), see examples below.

It is immediate that the matrix elements

\[
a_T(l, u; k, v) := \langle |k \otimes v\rangle |S^T|l \otimes u\rangle
\]

depend only on \( l \) and \( k \) only through \( l - k \), and vanish if \( l - k \) cannot be represented as a sum of \( T \) lattice vectors from the jump set \( j(C) \). In particular, the matrix elements vanish when \( l - k \) is outside of the scaled by \( T \) convex hull of the jump vectors \( j(C) \),

\[
P = P_j := \text{conv} \{j(e_k), k = 1, \ldots, c\}.
\]

We will use the shorthand

\[
A_T^{u,v}(k) := a_T(0, u; k, v) := \langle 0 \otimes v, S^T|k \otimes u\rangle
\]

for the amplitudes of the quantum random random walk starting at site \( 0 \) in internal state \( u \). Further, we denote by \( A_T(k) \) the corresponding operator \( H_0 \to H_k \), whose matrix coefficients are \( A_T^{u,v}(k) \).

We will be mostly interested in the probabilities

\[
p_T^{u,v}(k) := |A_T^{u,v}(k)|^2.
\]

It is clear that the unitarity of \( S \) implies that \( p_T^{u,v}(k, v) \) is a probability distribution on the basis \( \{|k, v\rangle\}_{k \in L, v \in \mathbb{C}^c} \) of \( H_L \) for any \( T \geq 0 \) and norm one \( u \in H \).

By construction, it is also clear that the amplitudes \( a_T(0, u; k, v) \) belong to the (dense) subspace of \( H_L \) of vectors with all almost all components zero.
1.2 Examples

In this section we will look at a few examples of QRWs in $d = 2$.

1.2.1 Hadamard Coin

A popular class of examples uses the Hadamard (or Grover) coins. For $c = 4$, it is given by

$$U = \frac{1}{2} \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

(2)

In the standard picture, the jumps are the steps to the neighboring sites on the 2-dimensional integer lattice, so that the jump map takes the basis vectors into $j(C) = \{(0, \pm 1), (\pm 1, 0)\}$.

In this case, the resulting amplitudes have asymptotic support localized in the circle inscribed into the diamond $P$, and has been thoroughly analyzed, see e.g. [BBBP11].

Switching to the jump map sending the basis to $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$ leads to essentially equivalent picture: the difference now is that the support of the amplitudes now is not the (shifted) even sublattice of $\mathbb{Z}^2$, but the whole lattice, and the footprint acquires drift: it is centered at $(1/2, 1/2)$.

The simulated amplitudes (or rather the corresponding probabilities averaged over possible initial internal state) are shown on the left display of Figure 3 - after 400 steps of the walk.

One can, of course, have completely different jump maps for the same coin. The remaining three displays (second through fourth, left-to-right) show the probabilities for the jump map sending the basis to

$\bullet$ $j_1(C) = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$;

$\bullet$ $j_2(C) = \{(0, 0), (2, 1), (1, 2), (1, 1)\}$;

$\bullet$ $j_3(C) = \{(0, 0), (2, 1), (1, 2), (2, 2)\}$ and

$\bullet$ $j_4(C) = \{(0, 0), (2, 0), (2, 1), (1, 0)\}$,

(left to right).

Figure 1: Probability distributions for QRWs with $d = 4$ Hadamard coin and jump vectors $j_1, \ldots, j_4$, as in section 1.2.1.
1.2.2 Random Unitary Coin

Below are the results of the similar simulations, with \( d = 4, T = 400 \) time steps, and the jump maps \( j_1, \ldots, j_4 \) shown above, for the (randomly generated) unitary matrix

\[
U = \begin{pmatrix}
-0.331759 + 0.069082i & 0.471768 + 0.231231i & -0.278617 - 0.583254i & -0.425926 + 0.099521i \\
0.368644 - 0.479381i & -0.113567 + 0.513171i & -0.443628 + 0.254345i & -0.218580 + 0.220861i \\
0.169821 - 0.199957i & 0.206809 - 0.447012i & 0.177488 + 0.271570i & -0.723490 - 0.244741i \\
-0.654156 + 0.150721i & -0.444209 + 0.088396i & -0.315095 + 0.340811i & -0.234176 - 0.271940i
\end{pmatrix}
\]

(3) \{eq:ucoin\}

Figure 2: Probability distributions for QRWs with \( d = 4 \), unitary coin (3) and jump maps \( j_1, \ldots, j_4 \), as in section 1.2.1.

{fig:hadamarrand}

2 Amplitudes as Oscillating Integrals

As simulations show, the coefficients \( A_T(\cdot) \) exhibit beautiful interference patterns, localized properly within the polytope \( tP \). At the lattice points inside the scaled by \( T \) convex hull \( P \) of the jump vectors but outside of the visible support of the distribution, the amplitudes in fact do not vanish but rather are exponentially small.

Let \( \overline{x} = (x_1, \ldots, x_d) \) be the vector of (for now, symbolic) variables. We associate to the amplitude vectors their generating (in general, Laurent) polynomials

\[
A_T^{u,v}(\overline{x}) := \sum_{k \in L} A_T^{u,v}(k)\overline{x}^k,
\]

where we use the shorthand

\[
\overline{x}^k = \prod_{l=1}^{d} x_l^{k_l}.
\]

We can consider \( A_T(\overline{x}) \) as matrices with coefficients in \( \mathbb{C}(\overline{x}) \), the ring of Laurent polynomials in variables \( x_1, \ldots, x_d \).

The following Proposition is a straightforward reformulation of the definitions:

**Proposition 2.1.** For any \( T \geq 0 \),

\[
A_T(\overline{x}) = M(\overline{x})^T,
\]

where \( M(\overline{x}) = \Delta(\overline{x})U \), and \( \Delta(\overline{x}) \) is the diagonal matrix with \( \Delta(\overline{x})_{l,l} = \overline{x}^{l(e_l)} \).
2.1 Complex Variables

From now on, we interpret all $x, z$ as complex variables.

Let $\mathbb{T}_L \cong \mathbb{R}^d/L^*$ be the $d$-dimensional torus (of characters on $L$). We can identify $\mathbb{T}_L$ with the collection of vectors $\bar{x} \in \mathbb{C}^d : |x_l| = 1, l = 1, \ldots, d$. The product of $\mathbb{T}_L$ with the unit circle is called the extended torus:

$$\mathbb{T} = \mathbb{T}_L \times \mathbb{T}^1 = \{ (\bar{x}, z) : \bar{x} \in \mathbb{T}_L, |z| = 1 \}.$$ 

Obviously, $M(\bar{x})$ is unitary for any $\bar{x} \in \mathbb{T}_L$.

The precise structure of matric coefficients of $A_T$ is best understood in terms of the spectral surface.

**Definition 2.2.** Consider $M(\bar{x})$ as the function on $\mathbb{T}_L$ with values in the unitary operators. Then

$$\Sigma := \{ (\bar{x}, z) : z \text{ is an eigenvalue of } M(\bar{x}) \}$$

is called the spectral surface.

The following is immediate:

**Lemma 2.3.** The spectral surface is (real) algebraic, and its projection $p : \Sigma \to \mathbb{T}_L$ is a $c$-fold branching covering (counted with multiplicities).

**Proof.** Indeed, the spectral surface is given by the zero set of the Laurent polynomial

$$\det(zI - M(\bar{x})), \quad \text{and the spectrum of the unitary matrix } M(\bar{x}) \text{ belong to the unit circle.}$$

We will denote the fiber of the projection to $\mathbb{T}_L$ as

$$\Sigma(\bar{x}) := p^{-1}(\bar{x}) \cap \Sigma = \{ z : (\bar{x}, z) \in \Sigma \}.$$

The spectral theorem implies that one can associate to any point $(\bar{x}, z)$ on the spectral surface the projector $P(\bar{x}, z)$ to the corresponding eigenspace, so that

$$M(\bar{x}) = \sum_{z \in \Sigma(\bar{x})} zP(\bar{x}, z).$$

These projectors $P(\bar{x}, z)$ are orthogonal for different $z \in \Sigma(\bar{x})$, and sum up to the unity,

$$\sum_{z \in \Sigma(\bar{x})} P(\bar{x}, z) = I.$$

Therefore,

$$A_T(\bar{x}) = \sum_{z \in \Sigma(\bar{x})} z^T P(\bar{x}, z). \quad \text{(4) \eq:amplitudes}$$

At the points where the spectral surface is smooth, $z$ can be represented (locally) as a function of $\bar{x}$. If we define $\text{Sing} = \text{Sing}(U) \subset \Sigma$ as the set of singular points of $\Sigma$, one has

**Proposition 2.4.** The set $S$ is a real algebraic subvariety of $\Sigma$, such that $\Sigma^\circ = \Sigma - \text{Sing}$ is everywhere dense in $\Sigma$, and such that the image of the projection of $\text{Sing}$ to $\mathbb{T}_L$ is nowhere dense.

Remark that the smoothness of the spectral surface at $(\bar{x}, z)$ does not imply necessarily that the rank of the projector $P(\bar{x}, z)$ at the point is 1. One can have a component in $\Sigma$ of multiplicity $> 1$. This does not happen, however, for a generic $U$. 


2.2 Amplitudes as Oscillating Integrals

To recover the matrix elements from the expression (4) we apply the standard Cauchy formula (or, equivalently, inverse the Fourier transform):

**Proposition 2.5.** The amplitudes \( A_T(k) \) are given by

\[
A_T(k) = \frac{1}{(2\pi i)^d} \int_{\mathbb{T}_L} \frac{e^{-k \cdot M(\xi)}}{\xi} d\xi = \frac{1}{(2\pi i)^d} \int_{\mathbb{T}_L \times \Sigma(\xi)} \sum_{\alpha} z^T \xi^{-k} P(\xi, z) d\xi. \quad (5)
\]

Introducing the logarithmic coordinates on the torus \(|\xi| = 1, l = 1, \ldots, d\),

\[
x_k = \exp(i \xi_k), z = \exp(i \xi),
\]

we obtain

\[
A_T(k) = \int_{0 \leq \xi_k \leq 2\pi} \sum_{\xi \in \Sigma(\xi)} e^{i T(\xi - (\rho, \xi))} P(\xi, \xi) d\xi, \quad (6) \quad \{\text{eq:osci}\}
\]

where we denote by \( \rho = k/T \) the rescaled index, and \( \xi = (\xi_1, \ldots, \xi_d) \). (We retain the notation \( \Sigma(\xi) \), \( P(\xi, \xi) \) etc for the logarithmic coordinates, where this does not lead to confusion.)

The identity (6) expresses the amplitudes as some special cases of oscillating integrals. Namely, let \( \text{Sing}_\xi = \rho(\text{Sing}) \) be the projection of the singular set of \( \Sigma \) to the \( \mathbb{T}_L \) (which is a nowhere dense, closed semialgebraic subset of \( \mathbb{T}_L \)). In a vicinity \( U \) of a point \( \xi_0 \in \mathbb{T}_L - \text{Sing}_\xi \), the branches of \( \xi \) can be represented locally as functions of \( \xi \), so that the corresponding contribution to the integral (6) becomes

\[
\sum_m \sum_{\zeta_0 \in \Sigma(\xi_0)} \int_U e^{i T(\zeta_a(\xi) - (\rho, \xi))} P_a(\xi(s)) s_m d\xi, \quad (7) \quad \{\text{eq:osc2}\}
\]

where we denote by \( P_a(\xi) = P(\xi, \xi_a) \), and the external summation is over open vicinities of an open covering \( \cup_m U_m = \mathbb{T}_L - \text{Sing}_\xi \), and \( s_m \) is a subordinated partition of the unity.

This representation allow one to use various tools from the oscillating integrals theory or from singularity theory, to explore the large \( T \) asymptotics.

Thus, a standard result implies that if for some \( \rho \), the phase \( \zeta_a(\xi) - (\rho, \xi) \) has no critical points for all branches \( \zeta_a \), the integral decays faster than any power of \( T \).

Therefore, for \( \rho \) not in the range of \( d\zeta_a \), for any \( a \), the amplitudes are decaying superpolynomially (in fact, exponentially fast) at the indices \( k \approx \rho T \), as \( T \rightarrow \infty \).

If the phase \( \zeta_a - (\rho, \xi) \) does have a critical point, it is, for a generic \( \rho \), a Morse one, i.e. has non-degenerate quadratic part. In this case, again according to the standard results, the amplitudes decay as \( T^{-d/2} \) (this meshes well with the fact that squared amplitudes behave generically like \( T^{-d} \), as they represent a discrete probability distribution supported by a subset of size \( \Theta(T^d) \)).

Further, near a typical point of the boundary of the essential support of the amplitudes, they are given by Airy integral. One can find the Pearcey integrals, and further oscillating integrals (depending on parameters).

3 Probability Measures associated with a QRW

Like the amplitudes, the (discrete) nonnegative measures

\[
p_T^{u,v}(k) = |A_T^{u,v}(k)|^2
\]

oscillate wildly for large \( T \). However, after rescaling they converge weakly to a well-defined probability measure. This probability measure has a nice description, that has been described first in [GJS04].
3.1 Gauss Map

For a smooth point \( (\xi, \zeta) \in \Sigma \) of the spectral surface, we denote by \( G(\xi, \zeta) \) the differential of \( \zeta(\xi) \), an implicit function parameterizing the corresponding branch nearby (we remark that all tangent spaces to points of \( \mathcal{T} = \mathbb{T}_L \times \mathbb{T} \) are canonically identified with \( \mathbb{R}^d \times \mathbb{R}^d \). We will be referring to \( G \) as the Gauss map. The Gauss map is defined on a dense subset of \( \Sigma \).

Fix \( u \in H \). We will denote by

\[
p_T^u(k) := \int_{|v|=1} p_T^u(T, v) \sigma(dv) = \sum_{\rho \in \mathcal{C}} p_T^u(T \rho, v)
\]

the total probability measure corresponding to initial state \( |0, u\rangle, |u| = 1 \).

The rescaled (discrete) probability measures \( \pi_T^u(\rho) := p_T^u(T \rho) \) are supported by \( \mathcal{P} \cap L/T \), the intersection of convex hull of the jump vectors with the rescaled lattice \( L/T \).

3.2 Weak Limits

The limiting behavior of these measures is given by the following theorem. A version of this theorem was proven in [GJS04]) using momenta, but I will sketch an alternative proof relying on the tools of the theory of oscillating integrals.

**Theorem 3.1.** Define the nonnegative densities \( s^u \) on (the dense nonsingular part of) the spectral surface \( \Sigma \) as

\[
s^u(\xi, \zeta) := \langle u, P(\xi, \zeta) u \rangle d\xi.
\]

Then, as \( T \to \infty \), \( \pi_T^u \) weakly converges to the push-forward \( G_* s^{u^T} \) of the density \( s^u \) under the Gauss map.

**Sketch of the proof.** We rely on the ideas of Duistermaat [Dui74].

To prove weak convergence of the probability measures, it is enough to prove the convergence of the integrals

\[
\int h(p) p_T^u(T \rho, v) d\rho \to \int h(p) G_* s^{u^T}(\rho)
\]

for smooth test functions \( h \).

We will rely on (7). Substituting, we obtain

\[
p_T^u(k, v) = |A_T^u(k)|^2 = \int \int \sum_{\zeta_a \in \Sigma(\xi, \zeta_a) \in \Sigma(\xi')} e^{i T (\zeta_a(\xi) - \zeta_a'(\xi) - (\rho, \xi - \xi'))} \langle P(\xi', \zeta_a') u, v \rangle \langle v, P(\xi, \zeta_a) u \rangle d\xi d\xi'.
\]

Summing \( \langle Au, v \rangle \langle v, Bu \rangle \) over \( v \) running through an orthonormal basis (or averaging over \( v \) in the unit sphere) results in \( \langle u, A^T B u \rangle \), so that

\[
p_T^u(k) = \int \int \sum_{\zeta_a \in \Sigma(\xi, \zeta_a) \in \Sigma(\xi')} e^{i T (\zeta_a(\xi) - \zeta_a'(\xi) - (\rho, \xi - \xi'))} \langle u, P(\xi', \zeta_a') P(\xi, \zeta_a) u \rangle d\xi d\xi'.
\]

If the sequence of the measures \( \pi_T^u(\rho) \) converges weakly to a probability measure (it does, at least along some subsequence, as all of these measures are supported on a compact \( \mathcal{P} \)), one has

\[
\lim_{T \to \infty} \sum h(T \rho) \pi_T^u(\rho) = \lim_{T \to \infty} T^d \int h(p) p_T^u(T \rho) d\rho =
\]

\[
\lim_{T \to \infty} T^d \int h(p) \sum_{\zeta_a \in \Sigma(\xi, \zeta_a) \in \Sigma(\xi')} e^{i T (\zeta_a(\xi) - \zeta_a'(\xi) - (\rho, \xi - \xi'))} \langle u, P(\xi', \zeta_a') P(\xi, \zeta_a) u \rangle d\xi d\xi' d\rho =
\]

\[
\lim_{T \to \infty} T^d \int h(p) \sum_{\zeta_a \in \Sigma(\xi, \zeta_a) \in \Sigma(\xi')} e^{i T (\zeta_a(\xi) - \zeta_a'(\xi) + (\rho, \eta))} \langle u, P(\xi', \zeta_a') P(\xi, \zeta_a) u \rangle d\xi d\eta d\rho
\]
Now, Duistermaat’s trick is to estimate the asymptotic Laplace inetgral in variables $\eta$ and $\rho$: as one can easily see, the restrictions of the phases
\begin{align*}
\zeta_{\alpha}(\xi) - \zeta_{\alpha'}(\xi + \eta) + (\rho, \eta)
\end{align*}
to the $2d$-dimensional spaces of constant $\xi$ have unique Morse critical point of index $d$ and determinant $1$ in each of the fibers, attained at $\eta = 0, \rho = d\zeta_{\alpha'}(\xi)$. Using the standard formulae of the Laplace integrals, and the fact that the projectors $P(\xi, \zeta)$ are orthogonal at distinct $\zeta \in \Sigma(\xi)$, we see that the integral
\begin{align*}
\lim_{T \to \infty} T^d \int \int \int h(\rho) \sum_{\zeta_a \in \Sigma(\xi), \zeta_{a'} \in \Sigma(\xi')} e^{iT(\zeta_a(\xi) - \zeta_{a'}(\xi + \rho, \eta))} \langle u, P(\xi', \zeta_{a'}) P(\xi, \zeta_a) u \rangle d\xi d\eta d\rho =
(2\pi)^{-d} \int \sum_{\zeta_a \in \Sigma(\xi)} h(d\zeta_a(\xi)) \langle u, P(\xi, \zeta_a) u \rangle d\rho,
\end{align*}
which is equivalent to the claim of the theorem. \qed

Averaging of the probability measures $\pi_T^u(\cdot)$ with respect to $u$ (again, either over an orthonormal baisi, or over the unit sphere), results probability measures $\pi_T$, again supported on $P$.

**Corollary 3.2.** The probability measures $\pi_T$ converge, weakly, to the push-forward of the density on the spectral surface equal to $d^{-1} \mathrm{rk}(R(\xi, \zeta)) d\xi$. 

## 4 Localization

Localization is a pattern in quantum random walks that attracted significant attention in the literature.

We use a somewhat generalized notion of localization in quantum random walks. Namely,

**Definition 4.1.** The quantum random walk exhibits strong localization if for some initial state $u$, there is a sequence of times $T_1, T_2, \ldots$ and states $|k_1, v_1), |k_2, v_2), |k_3, v_3), \ldots$ such that the sequence of probabilities has nonzero lower limit:
\begin{align*}
\liminf_{k \to \infty} p^u_{T_k}(k, v_k) > 0.
\end{align*}
If the sequence of vectors $k/T_k$ converges to a vector $s \in P$, then we say that there is localization at asymptotic speed $s$.

The quantum walk localizes weakly if the limiting measure $\pi_{u, U}$ defined in Theorem 3.1 has an atom.

In other words, we allow the particle to localize at some point that moves with linear speed, not necessarily equal to zero.

It is immediate that strong localization implies that the set of asymptotic speeds is nonempty (by compactness of $P$ and Tychonov theorem), and that strong localization implies weak localization.

In [KSY16] the equivalence of strong and weak localizations for a special class of one-dimensional quantum random walks was proven. In fact, this is always the case:

**Proposition 4.2.** For translation invariant quantum random walks on lattices, the strong and weak localizations are equivalent.

To prove this, we use the following corollary of Theorem 3.1.

Define the monomial torus in $\mathbb{T}$ the torus given by equation $z^m x^l = 1$, for some integer $m \neq 0, l \in \mathbb{Z}^d$.

**Proposition 4.3.** The quantum localizes weakly only if the spectral surface contains a monomial torus as a component.
Proof. Existence of an atom \( a \in P \) in the limiting measure \( \pi \) implies that the Gauss map \( G \) sends a set of positive measure \( A \subset \Sigma \) to a point. This implies that there is a point \( (\xi, \zeta) \in \Sigma^o \) on the smooth part of the spectral surface, such that \( A \) is dense at the point (that is the fraction of small ball around that point which is in \( A \) tends to 1 as the radius of the ball tends to zero).

By Fubini, for almost any \( v \in \mathbb{R}^d, |v| = 1 \), the curve \( (\xi(t), \zeta(t)) \in \Sigma^o, t \in (-\epsilon, \epsilon) \) such that \( \xi(t) = \xi_* + tv \) for small \( t \), the intersection of the curve with \( A \) is dense at \( t = 0 \), and therefore (by analyticity of the curve, and algebraicity of the Gauss map), the curve is in \( A \) for all \( t \). This implies that in some vicinity of \( (\xi, \zeta) \), the Gauss map is a constant, and, therefore, again by analyticity of \( \Sigma \), it is constant on an open component of \( \Sigma^o \). Thus this component is the level set of a monomial. \( \square \)

This implies, immediately,

**Corollary 4.4.** If a quantum random walk localizes weakly, the atom belongs to the intersection of sublattice \( \mathbb{Z}^d/k, 1 \leq k \leq c \) and the jump set convex hull \( P \).

Also, if the limiting probability measure \( \pi_{u,U} \) has an atom at \( s \in \mathbb{Z}^d/k \cap P \), then there is strong localization at the speed \( s \).

In other words, the speeds at which localization exists are all rational, with denominators bounded by the dimension of the chirality space.

## 5 Random Coin

Thus far we established that the limiting probability density of the rescaled position of a QRW corresponding to a coin \( U \) with the starting state \( |0 \otimes u\rangle, |u|^2 = 1 \), averaged over \( u \), is the image of the measure \( \xi \) lifted to \( \Sigma \) under the Gauss map.

We will denote this limiting probability measure corresponding to the coin \( U \) as \( \gamma_U \).

It is supported, for all coins \( U \), by the convex hull \( P \) of the jump vectors \( j_i = j(e_i), i = 1, \ldots, n \).

A natural question to ask is about the average behavior of the asymptotic measures \( \gamma_U \) for typical \( U \).

Namely, what is the average of the measures \( \gamma_U \) as \( U \) is distributed over \( SU(c) \) according to Haar measure?

The answer is surprisingly simple, and is given by the following theorem:

**Theorem 5.1.** The averaging of \( \gamma_U \) is the pushforward of the uniform probability measure on the simplex \( \Delta \) spanned by the basis vectors \( e_i, i = 1, \ldots, c \) under the jump map, \( j: e_i \mapsto j_i \), i.e.

\[
\int \gamma_U dU = \mu(\bigcup_{n=1}^{\Delta_\cap(C)})
\]

We start with a standard perturbation computation:

**Lemma 5.2.** If \( t \mapsto (\xi(t), \zeta(t)) \in \Sigma^o \) is a germ of a curve in the smooth part of the spectral surface, and the corresponding projector \( P(\xi(0), \zeta(0)) = |v\rangle\langle v| \) has rank 1, then

\[
\dot{\zeta} = \langle \Delta(\dot{\xi}) v, v \rangle,
\]

(here we denote by dot the derivative with respect to the parameter on the curve, and by \( \Delta(\cdot) \) the diagonal matrix with the corresponding vector on the diagonal).

**Proof.** Under the assumptions, one can choose the eigenvectors smoothly depending on \( t \). Recall that \( \vec{x} = \exp(i\xi); \vec{z} = \exp(i\zeta) \). Differentiating the identity

\[
zv = \Delta(\vec{x}) U v,
\]
and using the fact that $|v|^2 = 1 \Rightarrow \langle \dot{v}, v \rangle = 0$, we obtain

$$ \dot{z}v + z \dot{v} = \Delta(\hat{x}) \hat{U}v + \Delta(x) \hat{U} \dot{v}. $$

Next we contract this identity with $v$. Using the equalities

$$ \Delta(\hat{x}) \hat{U}v = \Delta(\hat{x}) \Delta(x)^{-1} \Delta(x) \hat{U}v = i z \Delta(\xi) v, $$

and

$$ \langle \Delta(x) \hat{U} \dot{v}, v \rangle = \langle \dot{v}, \Delta(x) \hat{U}^\dagger v \rangle = \langle \dot{v}, (\Delta(x) \hat{U})^{-1} v \rangle = \langle \dot{v}, z^{-1} v \rangle = 0, $$

we arrive at the desired identity.

**Corollary 5.3.** The differential of $\zeta$ as a function of $\xi$ is given by

$$ d\zeta = \sum_c |v_c|^2 \langle j(c), d\xi \rangle $$

**Proof.** Direct substitution.

In other words, the Gauss map at a smooth point of the spectral surface where the corresponding eigenspace has dimension 1 is given by the convex combination of the jump vectors, with weights equal to the squared amplitudes of the normalized eigenvector.

The proof of the Theorem 5.1 follows now from the standard facts:

**Proof of Theorem 5.1.** Consider the average over the coins of the sum of the images of the Gauss map at points of the spectral surface over $\xi$. For almost all coins, the spectral surface is smooth at those points, and the corresponding eigenspaces one-dimensional. Further, by the unitary invariance of the Haar measure, the distributions of those one-dimensional subspaces will be SU-invariant, and therefore the corresponding eigenvectors can be chosen to be uniformly distributed over the unit sphere. As is well-known, the vector of squared absolute values of coordinates (in any orthonormal basis) of a random vector uniformly distributed over the unit sphere is uniformly distributed in the standard simplex.

This proves that the average (over coins) of the images under the Gauss map of the points of the spectral surface in a given fiber are the push-forward under the jump map of the uniform measure on the standard simplex. As the result is independent of $\xi$, averaging over $\xi$ does not change the resulting density.

**5.1 Simulations**

Here are the results of the averaging of the probability density after $T = 40$ steps over 1000 randomly generated unitary coins, for the $d = 4$ and the jump maps corresponding to the examples of 1.2.

One can easily identify visually that the resulting density is the projection of the uniform measure under the corresponding jump map.

**References**


Figure 3: Averaged probability distributions for QRWs with jump vectors $j_1, \ldots, j_4$, as in section 1.2.1.


