Asymptotics of multivariate sequences in the presence of a lacuna [1]

Abstract: We explain a discontinuous drop in the exponential growth rate for certain multivariate generating functions at a critical parameter value, in even dimensions $d \geq 4$. This result depends on computations in the homology of the algebraic variety where the generating function has a pole. These computations are similar to, and inspired by, a thread of research in applications of complex algebraic geometry to hyperbolic PDEs, going back to Leray, Petrowski, Atiyah, Bott and Gärding. As a consequence, we give a topological explanation for certain asymptotic phenomenon appearing in the combinatorics and number theory literature. Furthermore, we show how to combine topological methods with symbolic algebraic computation to determine explicitly the dominant asymptotics for such multivariate generating functions. This in turn enables the rigorous determination of integer coefficients in the Morse-Smale complex, which are difficult to determine using direct geometric methods.

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1 Introduction

Let $k \geq 1$ be an integer and for P and Q coprime polynomials over the complex numbers let

$$F(\mathbf{z}) = \frac{P(\mathbf{z})}{Q(\mathbf{z})^k} = \sum_{\mathbf{r} \in \mathbb{Z}^d} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}} = \sum_{\mathbf{r} \in \mathbb{Z}^d} a_{\mathbf{r}} z_1^{r_1} \cdots z_d^{r_d} \tag{1.1}$$

be a rational Laurent series converging in some open domain $\mathcal{D} \subset \mathbb{C}^d$. The field of Analytic Combinatorics in Several Variables (ACSV) describes the asymptotic determination of the coefficients $a_{\mathbf{r}}$ via complex analytic methods. Let $\mathcal{V} = \mathcal{V}_Q$ denote the algebraic set $\{\mathbf{z} : Q(\mathbf{z}) = 0\}$ containing the singularities of $F(\mathbf{z})$. The methods of ACSV, summarized below, vary in complexity depending on the nature of $\mathcal{V}$. When $\mathcal{V}$ is smooth, that is when Q and ∇Q do not vanish simultaneously, explicit formulae may be obtained that are universal outside of cases when the curvature of $\mathcal{V}$ vanishes [PW02, PW13]. When $\mathcal{V}$ is the union of transversely intersecting smooth surfaces, similar residue formulae hold [PW04, PW13, BMP19b]. The next most difficult case is when $\mathcal{V}$ has an isolated singularity whose tangent cone is quadratic, locally of the form $x_1^2 - \sum_{j=2}^d x_j^2$. These points, satisfying the *cone point hypotheses* [BP11, Hypotheses 3.1], are called *cone points*; the necessary complex analysis in the vicinity of a cone point singularity, based on the work of [ABG70], is carried out in [BP11].

Let $|\mathbf{r}| = |r_1| + \cdots + |r_d|$. In each of these cases, asymptotics may be found of the form

$$a_{\mathbf{r}} \sim C(\hat{\mathbf{r}}) |\mathbf{r}|^\beta \mathbf{z}_*(\hat{\mathbf{r}})^{-\mathbf{r}} \tag{1.2}$$

where C and $\mathbf{z}_*$ depend continuously on the direction $\hat{\mathbf{r}} := \mathbf{r}/|\mathbf{r}|$. A very brief summary of the methodology is as follows. The multivariate Cauchy integral formula gives

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_T \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}}, \tag{1.3}$$

where $T \subseteq \mathcal{D}$ is a torus in the domain of convergence and $d\mathbf{z}/\mathbf{z}$ is the logarithmic holomorphic volume form $z_1^{-1} \cdots z_d^{-1} dz_1 \wedge \cdots \wedge dz_d$. Expand the chain of integration T so that it passes through the variety $\mathcal{V}$, touching it for the first time at a point $\mathbf{z}_*$ where the logarithmic gradient of Q is normal to $\mathcal{V}$, and continuing to at least a multiple $(1+\varepsilon)$ times this polyradius. Let $\mathcal{I}$ be the intersection with $\mathcal{V}$ swept out by the homotopy of the expanding torus. The residue theorem, described in Definition 3.4 below, says that the integral (1.3) is equal to the integral over the expanded torus plus the integral of a certain residue form over $\mathcal{I}$. Typically, $\mathbf{z}^{-\mathbf{r}}$ is maximized over $\mathcal{I}$ at $\mathbf{z}^*$, and integrating over $\mathcal{I}$ yields asymptotics of the form (1.2).

In the case of an isolated singularity with quadratic tangent cone, Theorem 3.7 of [BP11] gives such a formula but excludes the case where $d = 2m > 2k+1$ is an even integer and $d-1$ is greater than twice the power k in the denominator of (1.1). In that paper the asymptotic estimate obtained is only $a_{\mathbf{r}} = o(|\mathbf{r}|^{-m} \mathbf{z}_*^{-\mathbf{r}})$ for all m, due to the vanishing of a certain Fourier transform. This leaves open the question of what the correct asymptotics are and whether they are smaller by a factor exponential in $|\mathbf{r}|$.

In [BMPS18] it is shown via diagonal extraction that, for $k=1$ and a class of polynomials Q with an isolated quadratic cone singularity, in fact $a_{n,\ldots,n}$ has strictly smaller exponential order than $\mathbf{z}_*^{-(n,\ldots,n)}$. Diagonal extraction applies only to coefficients precisely on the diagonal, leaving open the question of behavior in a neighborhood of the diagonal[2], and leaving open the question of whether this behavior holds beyond the

[2] To see why this is important, consider the function $(x-y)/(1+x+y)$ that generates differences of binomial coefficients $\binom{i+j+1}{i} - \binom{i+j+1}{j}$; diagonal coefficients are zero but those nearby have order approaching $n^{-1/2} 4^n$.

particular class, for all polynomials with cone point singularities. The purpose of the present paper is to use ACSV methods to show that indeed the behavior is universal for cone points, that it holds in a neighborhood of the diagonal, and to give a topological explanation.

2 Main results and outline

Main result

Let F, P, Q and $\{a_{\mathbf{r}}\}$ be as in (1.1), choosing signs so that $Q(\mathbf{0}) > 0$. Throughout the paper we denote by $\mathbf{L} : \mathbb{C}_*^d \to \mathbb{R}^d$ the coordinatewise log-modulus map:

$$\mathbf{L}(\mathbf{z}) := \log |\mathbf{z}| = (\log |z_1|, \ldots, \log |z_d|) \, . \tag{2.1}$$

Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and let $\mathcal{M} := \mathbb{C}_*^d \setminus \mathcal{V}$ be the domain of holomorphy of $\mathbf{z}^{-\mathbf{r}} F(\mathbf{z})$ for sufficiently large $\mathbf{r}$. Let amoeba denote the amoeba of Q, defined by $\mathsf{amoeba} := \{\mathbf{L}(\mathbf{z}) : \mathbf{z} \in \mathcal{V}\}$. It is known [GKZ94] that the components of the complement of the amoeba are convex and correspond to Laurent series expansions for F, each component being a logarithmic domain of convergence for one series expansion. Let B denote the component of amoeba^c such that the given series $\sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ converges whenever $\mathbf{z} = \exp(\mathbf{x} + i\mathbf{y})$ with $\mathbf{x} \in B$.

We refer to the torus $T(\mathbf{x}) := \mathbf{L}^{-1}(\mathbf{x})$ as the **torus over x**. For any $\mathbf{r} \in \mathbb{R}^d$ we denote $\hat{\mathbf{r}} := \mathbf{r}/|\mathbf{r}|$ and

$$h_{\hat{\mathbf{r}}} := -\sum_{j=1}^{d} \hat{r}_j \log |z_j| \, .$$

For a subset $A \subset \mathbb{C}^d$, when $\hat{\mathbf{r}}$ and $\mathbf{z}_*$ are understood, we use the shorthand

$$A(-\varepsilon) := A \cap \{\mathbf{z} : h_{\hat{\mathbf{r}}}(\mathbf{z}) < h_{\hat{\mathbf{r}}}(\mathbf{z}_*) - \varepsilon\} \, . \tag{2.2}$$

Assume that $\mathcal{V}$ intersects the torus $\{\exp(\mathbf{x}_* + i\mathbf{y}) : \mathbf{y} \in (\mathbb{R}/(2\pi))^d\}$ at the unique point $\mathbf{z}_* = \exp(\mathbf{x}_*)$. We will be dealing with the situation when $\mathcal{V}$ has a quadratic singularity at $\mathbf{z}_*$. More specifically, we will assume that Q has a real hyperbolic singularity at $\mathbf{z}_*$.

Definition 2.1 (quadric singularity)**.** *We say that Q has a* **real hyperbolic quadratic singularity** *at $\mathbf{z}_*$ if $Q(\mathbf{z}_*) = 0, dQ(\mathbf{z}_*) = 0$ and the quadratic part q_2 of $Q = q_2(\mathbf{z}) + q_3(\mathbf{z}) + \ldots$ at $\mathbf{z}_*$ is a real quadratic form of signature $(1, d-1)$; in other words, there exists a real linear coordinate change so that $q_2(\mathbf{u}) = u_d^2 - \sum_{j=1}^{d-1} u_j^2 + O(|\mathbf{u}|^3)$ for $\mathbf{u}$ a local coordinate centered at $\mathbf{z}_*$.*

We denote by $T_{\mathbf{x}_*}(B)$ the open tangent cone in $\mathbb{R}^d$ to the component B of $\mathsf{amoeba}(Q)^c$, namely all vectors $\mathbf{v}$ at $\mathbf{x}_* := \mathbf{L}(\mathbf{z}_*)$ such that $\mathbf{x}_* + \varepsilon \mathbf{v} \in B$ for sufficiently small ε. The inequality defining $T_{\mathbf{x}_*}(B)$ is the same as the inequality $\widetilde{Q}(\mathbf{v}) > 0$ where $\widetilde{Q}$ is the homogenization (the quadratic term) of $Q(\exp(\mathbf{x}_* + \mathbf{v} + i\mathbf{y}_*))$, along with an inequality specifying $T_{\mathbf{x}_*}(B)$ rather than $-T_{\mathbf{x}_*}(B)$.

Definition 2.2 (tangent cone; supporting vector)**.** *The vector $\mathbf{r}$ is said to be* **supporting** *at $\mathbf{z}_*$ if $h_{\mathbf{r}}$ attains its maximum on the closure of B at $\mathbf{x}_*$, and if $\{dh_{\mathbf{r}} = 0\}$ intersects the tangent cone $T_{\mathbf{x}_*}B$ only at the origin. The open convex cone of supporting vectors is denoted $\mathcal{N}$ and the set of unit vectors over which it is a cone is denoted $\widehat{\mathcal{N}}$.*

Theorem 2.3 (main theorem)**.** *Let P be holomorphic in $\mathbb{C}^d$, Q a Laurent polynomial, k a nonnegative integer, B a component in the complement of the amoeba of Q, and $\sum_{\mathbf{r}\in E} a_\mathbf{r}\mathbf{z}^\mathbf{r}$ the corresponding Laurent series expansion of $F = P/Q^k$.*

Suppose that Q has real hyperbolic quadratic singularity at $\mathbf{z}_* = \exp(\mathbf{x}_*)$ such that $\mathbf{x}_*$ belongs to the boundary of B, and $\mathbf{z}_*$ is the unique intersection of the torus $\mathbf{T}(\mathbf{x}_*)$ with $\mathcal{V}$.

Let $K \subseteq \widehat{\mathcal{N}}$ be a compact set. Then, if d is even and $2k < d$,

(i) If $\varepsilon > 0$ is small enough then for any $\hat{\mathbf{r}} \in K$ there exists a compact cycle $\Gamma(\hat{\mathbf{r}})$, of volume uniformly bounded in $\hat{\mathbf{r}}$, such that for all $\hat{\mathbf{r}} \in K$ the cycle $\Gamma(\hat{\mathbf{r}})$ is supported by $\mathcal{M}(-\varepsilon)$ and

$$a_\mathbf{r} = \int_{\Gamma(\hat{\mathbf{r}})} \mathbf{z}^{-\mathbf{r}} \frac{P}{Q^k}\frac{d\mathbf{z}}{\mathbf{z}}. \tag{2.3}$$

(ii) If P is a polynomial, then

$$a_\mathbf{r} = \int_{\gamma(\hat{\mathbf{r}})} \mathrm{Res}_{\mathcal{V}} \mathbf{z}^{-\mathbf{r}} \frac{P}{Q^k}\frac{d\mathbf{z}}{\mathbf{z}} \tag{2.4}$$

for all but finitely many $\mathbf{r} \in E$. Here $\gamma(\hat{\mathbf{r}})$ is a compact $(d-1)$ cycle in $\mathcal{V}(-\varepsilon)$, of volume uniformly bounded in $\hat{\mathbf{r}}$, and $\mathrm{Res}_{\mathcal{V}}$ is the residue operator defined below in Section 3.

The heuristic meaning of this result is that, for purposes of computing the Cauchy integral, the chain of integration in (1.3) can be slipped below the height $h_\mathbf{r}(\mathbf{z}_*)$ at the singular point.

Motivation

Our motivating example for this theorem comes from the Gillis-Reznick-Zeilberger family of generating functions [GRZ83], discussed in [BMPS18, Theorems 9 – 12].

Example 2.4 (GRZ function at criticality)**.** *In four variables, let $F(\mathbf{z}) := 1/(1-z_1-z_2-z_3-z_4+27z_1z_2z_3z_4)$. When 27 is replaced by a parameter λ, it is shown in [BMPS18] via ACSV results for smooth functions that the exponential growth rate on the diagonal $|a_{n,n,n,n}|^{1/n}$ is a function of λ that approaches 81 as $\lambda \to 27$. At the critical value 27, however, the denominator Q of F has a real hyperbolic quadric singularity at $\mathbf{z}_* := (1/3, 1/3, 1/3, 1/3)$. Theorem 2.3 has the immediate consequence that the exponential growth of $a_\mathbf{r}$ for $\hat{\mathbf{r}}$ in a neighborhood of the diagonal is strictly less than that of $\mathbf{z}_*^{-\mathbf{r}}$, namely $81^{|\mathbf{r}|}$. Thus there is a drop in the exponential rate at criticality.*

With a little further work, understanding of the drop can be sharpened considerably. In Section 8 we state a result for general functions satisfying the conditions of Theorem 2.3. The result, Theorem 8.1, sharpens Theorem 2.3, quantifying the exponential drop by pushing the contour Γ down all the way to the next critical value. It is a direct consequence of Theorem 2.3 together with a deformation result of [BMP19a]. In the case of the GKZ function with at criticality, the following explicit asymptotics may be computed.

Theorem 2.5. *The diagonal coefficients $a_{n,n,n,n}$ of the function in Example 2.4 have an asymptotic expansion in decreasing powers of n, beginning as follows.*

$$a_{n,n,n,n} = 3\cdot\left(\frac{\left(4i\sqrt{2}-7\right)^n}{n^{3/2}}\,\frac{\left(5i-\sqrt{2}\right)\sqrt{-2i\sqrt{2}-8}}{24\pi^{3/2}} + \frac{\left(-4i\sqrt{2}-7\right)^n}{n^{3/2}}\,\frac{\left(-5i-\sqrt{2}\right)\sqrt{2i\sqrt{2}-8}}{24\pi^{3/2}}\right) \tag{2.5}$$

$$+O\left(9^n\, n^{-5/2}\right),$$

More generally, as $\mathbf{r} \to \infty$ *and* $\hat{\mathbf{r}}$ *varies over some compact neighborhood of the diagonal, there is a uniform estimate*

$$a_{\mathbf{r}} = 9^n\, n^{-3/2}\, 3C_{\hat{\mathbf{r}}} \cos(n\alpha_{\hat{\mathbf{r}}} + \beta_{\hat{\mathbf{r}}}) + O(9^n n^{-5/3})\,.$$

When $\hat{\mathbf{r}}$ *is on the diagonal, the constants* $C_{\hat{\mathbf{r}}}, \alpha_{\hat{\mathbf{r}}}$ *and* $\beta_{\hat{\mathbf{r}}}$ *specialize to produce* (2.5).

Heuristic argument

The plan is to expand the torus $\mathbf{T}$ via a homotopy $\mathbf{H}$ that takes it through the point $\mathbf{z}_*$ and beyond. Let $\mathcal{V}_*$ denote $\mathcal{V} \cap \mathbb{C}_*^d$, that is, the points of $\mathcal{V}$ with no coordinate vanishing. A classical construction, due to Leray, Thom and others, shows that $\mathbf{T}$ is homologous in $H_d(\mathcal{M})$ to a cycle Γ' which coincides above height $h(\mathbf{z}_*) - \varepsilon$ with a tube around a cycle σ; the height $h_{\mathbf{r}}$ is maximized on σ at the point $\mathbf{z}_*$ and the chain σ is the intersection of $\mathbf{H}$ with $\mathcal{V}_*$. We would like to see that σ is homologous to a class supported on $\mathcal{V}^{-\varepsilon}$.

To do this, we compute the intersection σ directly in coordinates suggested by the hypotheses of the theorem. In particular, we use local coordinates where, after taking logarithms, $\mathcal{V}$ is the cone $\{z_1^2 - \sum_{j=2}^d z_j^2 = 0\}$, and select a homotopy $\mathbf{H}$ from $\mathbf{x} + i\mathbb{R}^d$ to $\mathbf{x}' + i\mathbb{R}^d$ with $\mathbf{x} \in B$ so that the line segment $\overline{\mathbf{x}\mathbf{x}'}$ is perpendicularly bisected by the support hyperplane to B at $\mathbf{x}_*$. In these coordinates, the intersection class $\mathcal{I}$ is the cone $\{i\mathbf{y} : \mathbf{y} \in \mathbb{R}^d \text{ and } y_1^2 = \sum_{j=2}^d y_j^2\}$. The residue is singular at the origin (in new coordinates) but converges when $d > 2k+1$. Inside the variety $\mathcal{V}$, the cone $\mathcal{I}$ may be folded down so as to double cover the cone $\{x + i\mathbf{y} : \mathbf{y} \in \mathbb{R}^d, x > 0, y_1 = 0 \text{ and } x^2 = |\mathbf{y}|^2\}$; see Figure 1. The two covering maps have opposite orientations when d is even. The critical points of $h_{\mathbf{r}}$ restricted to $\mathcal{V}$ are obstructions for deforming the contour of integration downwards, and in this case the residue integral vanishes and the contour may be further deformed until it encounters the next highest critical point.

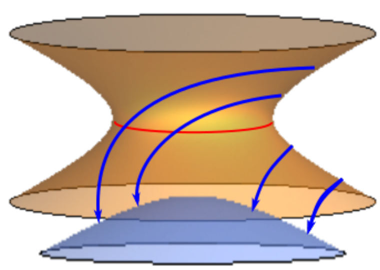

Figure 1: folding the cone down in two opposite directions

Outline of actual proof

The proof cannot precisely follow the heuristic argument because the intersection cycle construction and the residue integral theorem work only when $\mathcal{V}_*$ is smooth. We remark that the same trouble arose in the setting of [BP11]. There, we chose to adopt the method of [ABG70] to reduce the local integration cycle to its projectivized, compact counterpart: the so-called Petrowski or Leray cycles. That path required significant investment into analytic auxiliary results and, more importantly, would not immediately prove that the integration cycle in the presence of lacuna (i.e., when d is even and the denominator degree not too high) allows one to "slide" the integration cycle below the height of the cone point.

Thus we use a different strategy, first perturbing the denominator so that the perturbed varieties on which $Q(\mathbf{z}) = c$ for small c become smooth. (This kind of regularization also has the advantage, compared to what was used in [BP11], that we obtain information about the behavior of coefficients of the generating functions $P/(Q-c)^k$.) Then we study the behavior of the coefficients of the resulting generating functions as $c \to 0$. We denote the zero set of $Q(\mathbf{z}) - c$ by $\mathcal{V}_c$, write $(\mathcal{V}_c)_*$ for the points of $\mathcal{V}_c$ with non-zero coordinates, and denote the restriction of $\mathcal{V}_c$ to its points of height at most $h(\mathbf{z}_*) - \varepsilon$ by $\mathcal{V}_c(\leq -\varepsilon)$. It is easiest to work in the lower dimensional setting, with σ_c on $(\mathcal{V}_c)_*$ rather than Γ_c on $\mathcal{M}_c$, and to work in relative homology of $(\mathcal{V}_c)_*$ with respect to $\mathcal{V}_c(\leq -\varepsilon)$.

Section 4 lays the groundwork by computing the explicit intersection cycle in a limiting case of the perturbed variety as $c \downarrow 0$; this is a rescaled limit, and is smooth, in contrast to the variety at $c = 0$. Although the results of Section 4 are subsumed by later arguments, its focus on explicit computation allows for valuable intuition and visualization. Properties of our family of perturbations are given in Section 5. Section 6 uses this to complete a relative homology computation in $\mathcal{M}_c$ for sufficiently small c. It turns out that σ_c is not null-homologous in $H_{d-1}((\mathcal{V}_c)_*, \mathcal{V}_c(\leq -\varepsilon))$ but is instead homologous to an absolute cycle S_c which is homeomorphic to a $(d-2)$-sphere and lies in a small neighborhood of $\mathbf{z}_*$. Under the hypotheses of Theorem 2.3, the integral over S_c is easily seen to converge to zero as $c \downarrow 0$. The remaining outline of the proof is as follows.

1. Show that $\sigma_c \cong S_c$ in $H_{d-1}((\mathcal{V}_c)_*, \mathcal{V}_c(\leq -\varepsilon))$. This is accomplished in Section 6.
2. Pass to a tubular neighborhood to see that T in (1.3) may be replaced by the sum of tubular neighborhoods of S_c and a second chain γ, not depending on c, whose maximum height is at most $h(\mathbf{z}_*) - \varepsilon$. This is accomplished in Section 7.
3. Dimension analysis shows that the integral over the tubular neighborhood of S_c goes to zero as $c \downarrow 0$. This is accomplished in Section 7, proving Theorem 2.3.
4. Further Morse theoretic analysis shows that the contour γ is some integer times the sum of two standard saddle point contours. In Section 8 we show that for our motivating example this integer is in fact 3, proving Theorem 2.5

3 Preliminaries: tubes, intersection class, residue form

We recall some topological facts from various sources, most of which are summarized for application to ACSV in [PW13, pages 334 – 338]. Let K be any compact subset of $\mathcal{V}_*$ on which ∇Q does not vanish. The well known Tubular Neighborhood Theorem (for example, [MS74, Theorem 11.1]) states the following.

Proposition 3.1 (Tubular Neighborhood Theorem)**.** *The normal bundle over K is trivial and there is a global product structure of a tubular neighborhood of $\mathcal{V}_*$ in $\mathbb{C}_*^d$.* $\square$

This implies the existence of operators $\bullet$ and $\circ$, respectively the product with a small disk and with its boundary, mapping k-chains in $\mathcal{V}_*$ respectively to $(k+2)$-chains in $\mathcal{M}$ and $(k+1)$-chains in $\mathcal{M}$, well defined up to a natural homeomorphism as long as the radius of the disk is sufficiently small. We refer to $\circ\gamma$ as the **tube around** γ and $\bullet\gamma$ as the **tubular neighborhood** of γ. Elementary rules for boundaries of products imply

$$\begin{aligned}\partial(\circ\gamma) &= \circ(\partial\gamma)\,;\\ \partial(\bullet\gamma) &= \circ\gamma \cup \bullet(\partial\gamma)\,.\end{aligned} \tag{3.1}$$

Because $\circ$ commutes with ∂, cycles map to cycles, boundaries map to boundaries, and the map $\circ$ on the singular chain complex of $\mathcal{V}_*$ induces a map on homology $\circ : H_*(\mathcal{V}_*) \to H_*(\mathbb{C}_*^d \setminus \mathcal{V})$. This allows one to construct the **intersection class**, as in [BMP19a, Proposition 2.9].

Definition 3.2 (intersection class)**.** *Suppose Q vanishes on a smooth variety $\mathcal{V}$ and let $\mathbf{T}$ and $\mathbf{T}'$ be two d-cycles in $\mathcal{M}$ that are homologous in $\mathbb{C}_*^d$. Then there exists a unique class $\mathcal{I} = \mathcal{I}(\mathbf{T}, \mathbf{T}') \in H_{d-1}(\mathcal{V}_*)$ such that*

$$[\mathbf{T}] - [\mathbf{T}'] = \circ\mathcal{I} \qquad \text{in } H_d(\mathcal{M})\,.$$

The class $\mathcal{I}$ can be represented by the manifold $\mathbf{H} \cap \mathcal{V}$ for any manifold $\mathbf{H}$ with boundary $\mathbf{T} - \mathbf{T}'$ in $\mathbb{C}_*^d$ that intersects $\mathcal{V}$ transversely, with appropriate orientation (or, alternatively, by the image of the fundamental class of $\mathbf{H} \cap \mathcal{V}$ under the natural embedding).

We remark that if $\mathcal{V}$ is not smooth but its singularities (the locus where $Q = dQ = 0$) have real dimension less than $d-2$ then $\mathbf{H}$ generically avoids the singularities of $\mathcal{V}$, so $\mathcal{I}(\mathbf{T}, \mathbf{T}')$ is well defined. Although the singular set does not generically satisfy this dimensional condition, it does so in our applications, where the singular set is zero dimensional.

For our purposes, the natural cycles to consider are the tori $\mathbf{T}(\mathbf{x})$ for $\mathbf{x}$ in the complement of the amoeba of Q. In this case, there is an especially convenient choice of cobordism between $\mathbf{T}(\mathbf{x})$ and $\mathbf{T}(\mathbf{x}')$, namely the $\mathbf{L}$-preimage of the straight segment connecting $\mathbf{x}$ and $\mathbf{x}'$ (or its small perturbation). We will be referring to this cobordism as the *standard* one.

What are the good choices of $\mathbf{x}'$? We would like to make the integrand $F(\mathbf{z})\mathbf{z}^{-\mathbf{r}}d\mathbf{z}/\mathbf{z}$ exponentially small in $|\mathbf{r}|$ when $\mathbf{L}(\mathbf{z}) = \mathbf{x}'$, which happens if we can take $-\mathbf{r} \cdot \mathbf{x}'$ to have arbitrarily small modulus. When Q is a Laurent polynomial the feasibility of this follows from known facts about cones of hyperbolicity, as we now demonstrate.

First, recall that the Newton polytope of Q is the convex hull of the exponents $\mathbf{m}$ of the monomials of Q,

$$N(Q) = \mathsf{conv}(\{\mathbf{m} : q_\mathbf{m} \neq 0, Q(\mathbf{z}) = \sum_\mathbf{m} q_\mathbf{m}\mathbf{z}^\mathbf{m}\}) \subset \mathbb{R}^d. \tag{3.2}$$

The Newton polytope has vertices in the integer lattice, and the convex open components of the amoeba complement $\mathsf{amoeba}(Q)^c$ map injectively into the integer points in $N(Q)$ (see [FPT00]). Moreover, any vertex of $N(Q)$ has a preimage under this mapping, which is an unbounded component of $\mathsf{amoeba}(Q)^c$. The recession cone of the component $B_\mathbf{m}$ corresponding to a vertex $\mathbf{m}$ is the interior of the normal cone to $N(Q)$ at $\mathbf{m}$ (i.e., the collection of vectors $\mathbf{d}$ such that $\max_{\mathbf{r} \in N(Q)}(\mathbf{d}, \mathbf{r})$ is uniquely attained at $\mathbf{m}$). Notice that this normal cone is dual to the cone $N(Q)_\mathbf{m}$ spanned by $N(Q) - \mathbf{m}$.

Now, let B be the component of $\mathsf{amoeba}(Q)^c$ corresponding to the Laurent expansion of F under consideration, and let $\mathbf{m}$ be the corresponding integer vector in $N(Q)$. The vectors supporting B at $\mathbf{x}_*$ form an open cone contained in $N(Q)_\mathbf{m}$. Pick a generic $\mathbf{d}$ in the recession cone of B; then when $t > 0$ is large enough $\mathbf{x}_* - t\mathbf{d}$ is contained in an unbounded component B' of the complement to amoeba (this follows from the fact that the union of the recession cones of the unbounded components of $\mathsf{amoeba}(Q)^c$ are the complement to the set of functionals attaining their maxima on $N(Q)$ at multiple points, a positive codimension fan in $\mathbb{R}^d$). Hence, choosing $\mathbf{x}'$ in this component B' allows one to deform $\mathbf{T}' = \mathbf{T}(\mathbf{x}')$ while avoiding $\mathcal{V}$ so that $h_\mathbf{r}$ becomes arbitrarily close to $-\infty$ for all $\mathbf{r} \in K$.

Definition 3.3. *We will be referring to this component B' whose recession cone contains a vector $-\mathbf{d}$, for $\mathbf{d}$ in the recession cone of B, as* descending *with respect to the component B. Components B' with this property are in general not unique, but any choice of B' works for our argument.*

The following result is well known; see, e.g. [BMP19a, Proposition 2.14].

Definition 3.4 (residue form)**.** *There is a homomorphism $\mathrm{Res} : H^d(\mathcal{M}) \to H^{d-1}(\mathcal{V}_*)$ in deRham cohomologies such that for any class $\gamma \in H_d(\mathcal{V})$,*

$$\int_{\circ\gamma} \omega = \int_\gamma \mathrm{Res}\,(\omega)\,. \tag{3.3}$$

In general $\mathrm{Res}\,\omega$ can be derived locally from a form representing ω (we will be using Res also for the corresponding operators on differential forms). When $F = P/Q$ is rational with Q squarefree, Res commutes with multiplication by any locally holomorphic function and satisfies

$$Q \wedge \mathrm{Res}\,(F\, d\mathbf{z}) = P\, d\mathbf{z}\,.$$

More generally, if $F = P/Q^k$, then (see, e.g. [Pha11]) the residue can be expressed in coordinates as

$$\mathrm{Res}\,\left[\mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}}\right] := \frac{1}{(k-1)!} \frac{d^{k-1}}{dc^{k-1}} \left[\frac{P\mathbf{z}^{-\mathbf{r}}}{\mathbf{z}}\right] d\sigma\,, \tag{3.4}$$

where σ is the natural area form on $\mathcal{V}$ (characterized by $dQ \wedge \sigma = d\mathbf{z}$), and the partial derivatives with respect to c are taken in the coordinates where c is one of the variables.

Putting this together with the definition and construction of the intersection class and Cauchy's integral formula yields the following representation of the coefficients $a_\mathbf{r}$.

Proposition 3.5. *Suppose $F = G/Q^k = \sum_{\mathbf{r}\in E} a_\mathbf{r} \mathbf{z}^\mathbf{r}$ with G holomorphic and Q a polynomial, the series converging when $\log|\mathbf{z}|$ is in the component B of $\mathsf{amoeba}(Q)^c$. Let $\mathbf{x} \in B$ and $\mathbf{T}(\mathbf{x}) := \mathbf{L}^{-1}(\mathbf{x})$ be the torus with log-polyradius $\mathbf{x}$. Let $\mathbf{x}'$ be any other point in $\mathsf{amoeba}(Q)^c$. Then*

$$(2\pi i)^d\, a_\mathbf{r} \quad = \quad \int_{\mathcal{I}(\mathbf{T}(\mathbf{x}),\mathbf{T}(\mathbf{y}))} \mathrm{Res}\,\left[\mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}}\right] \; + \; \int_{\mathbf{T}(\mathbf{x}')} \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}}\,. \tag{3.5}$$

Moreover, if $\mathbf{y}$ is in B', a descending component with respect to B, and G is a polynomial, then for all but finitely many $\mathbf{r} \in E$,

$$a_\mathbf{r} = \frac{1}{(2\pi i)^d} \int_{\mathcal{I}(\mathbf{T}(\mathbf{x}),\mathbf{T}(\mathbf{x}'))} \mathbf{z}^{-\mathbf{r}} \mathrm{Res}\,\left(F(\mathbf{z})\,\frac{d\mathbf{z}}{\mathbf{z}}\right)\,.$$

Proof: The first identity is Cauchy's integral formula, the definition of the intersection class, and (3.3). The second identity follows from the fact that $\sup_{\mathbf{T}(\mathbf{x}')} |G/Q^k|$ and the volume of $\mathbf{T}(\mathbf{x}')$ grow at most polynomially in $|\mathbf{x}'|$ in $|\mathbf{x}'|$ on the torus over $\mathbf{x}'$. For $\mathbf{r} \in N(Q)_\mathbf{m}$ large enough in size, the degree of the decay of $|\mathbf{z}^{-\mathbf{r}}|$ overtakes that polynomial growth, so that the last term in (3.5) can be made arbitrarily small. As it is independent of $\mathbf{x}'$ as long as $\mathbf{x}'$ varies in the same component B', it vanishes identically. □

4 The limiting quadric

In this section we focus on the properties of a particular smooth quadratic function, namely

$$q(\mathbf{z}) := -1 + z_1^2 - \sum_{j=2}^{d} z_j^2 \,. \tag{4.1}$$

Let $\tilde{\mathcal{V}}$ denote the zero set of q. This quadric has the constant term -1 which does not appear in Q. The relation between $\mathcal{V}_c$ and $\tilde{\mathcal{V}}$ is that $\tilde{\mathcal{V}}$ is a rescaled limit of $\mathcal{V}_c$ as $c \downarrow 0$, blown up by $c^{-1/2}$. Our first statement deals with the gradient-like flow on $\tilde{\mathcal{V}}$ with respect to the function $h := x_0$.

Lemma 4.1. *The function h has two critical points $\mathbf{z}_\pm = (\pm 1, 0, \ldots, 0)$ on $\tilde{\mathcal{V}}$, both of index $d-1$. The stable manifold for $\mathbf{z}_+$ is the unit sphere*

$$\mathtt{S} := \{x_1^2 + \sum_{k=2} y_k = 0; y_1 = x_2 = \ldots = x_d = 0\}$$

and its unstable manifold is the upper lobe of the 2 sheeted real hyperboloid

$$\mathtt{H}_+ := \tilde{\mathcal{V}} \cap \mathbb{R}^d = \{x_1^2 - \sum_{k=2}^{d} x_k^2; x_1 > 0; \mathbf{y} = 0\} \,.$$

The stable manifold for $\mathbf{z}_-$ is the lower lobe $\mathtt{H}_-$ of this hyperboloid, while the unstable manifold of $\mathbf{z}_-$ is still the sphere $\mathtt{S}$.

Proof: The critical points can be found by a direct computation. Their indices are necessarily $d-1$, as h is the real part of a holomorphic function on a complex manifold [GH78]. Similarly, direct computation shows that the tangent spaces to $\mathtt{S}, \mathtt{H}_\pm$ are the stable/unstable eigenspaces for the Hessian matrices of h restricted to $\tilde{\mathcal{V}}$ at the critical points. Lastly, as the gradient vector field is invariant with respect to symmetries $\mathbf{y} \mapsto -\mathbf{y}$ and $(x_1, x_2, \ldots, x_d) \mapsto (x_1, -x_2, \ldots, -x_d)$, leaving $\mathtt{H}_\pm$ and $\mathtt{S}$ invariant, they are the invariant manifolds for the gradient flow. $\square$

Let $\mathbf{\Phi} : \mathbb{R}^d \times [-1, 1] \to \mathbb{C}^d$ be the homotopy defined by $\mathbf{\Phi}(\mathbf{y}, t) := t\mathbf{e}_1 + i\mathbf{y}$ (we use a new symbol because $\mathbf{H}$ is in principle only a cobordism). Let $\tilde{h}$ denote the height function $\tilde{h}(\mathbf{z}) = -\Re\{z_1\}$.

Theorem 4.2. *The intersection cycle of the homotopy $\mathbf{\Phi}$ with the variety $\tilde{\mathcal{V}}$ is the union of a hyperboloid $\mathtt{H}$ and a $(d-1)$-sphere $\mathtt{S}$ that intersect in a $(d-2)$-sphere $\mathcal{S}'$. These are given by equations (4.6) – (4.8). The orientation of the intersection cycle is continuous on each of the four smooth pieces, namely the upper and lower half of $\mathtt{H} \setminus \mathcal{S}'$ and the northern and southern hemispheres of $\mathtt{S} \setminus \mathcal{S}'$, but change signs when crossing $\mathcal{S}'$.*

Proof: Writing $\mathbf{z}_j := x_j + iy_j$, the equations for $\mathbf{z}$ such that $\mathbf{z}$ is in the range of $\mathbf{\Phi}$ and $q(\mathbf{z}) = 0$ become

$$|x_1| \leq 1 \tag{4.2}$$

$$x_j = 0 \qquad (2 \leq j \leq d) \tag{4.3}$$

$$x_1^2 - y_1^2 = 1 - \sum_{j=2}^{d} y_j^2 \tag{4.4}$$

$$x_1 y_1 = 0 \,. \tag{4.5}$$

Clearly this is the union of two sets, one obtained by solving (4.2) – (4.4) when $x_1 = 0$ and the other by solving (4.2) – (4.4) when $y_1 = 0$; these intersect along the solution to (4.2) – (4.4) when $x_1 = y_1 = 0$. The first of these is the one-sheeted hyperboloid $\mathtt{H} \subseteq i\mathbb{R}^d$ given by

$$-y_1^2 = 1 - \sum_{j=2}^d y_j^2 \,. \tag{4.6}$$

The second is the sphere $\mathtt{S} \subseteq \mathbb{R} \times i(\mathbb{R}^{d-1})$ given by

$$x_1^2 + \sum_{j=2}^d y_j^2 = 1 \,. \tag{4.7}$$

These intersect at the equator of the sphere $\mathtt{S}$, which is the neck of the hyperboloid $\mathtt{H}$. The intersection set is the sphere $\mathcal{S}'$ in $\{0\} \times i\mathbb{R}^{d-1}$ given by

$$\sum_{j=2}^d y_j^2 = 1 \,. \tag{4.8}$$

The intersection class is given by the intersection of $\tilde{\mathcal{V}}$ with any homotopy intersecting it transversely. While $\mathbf{\Phi}$ does not intersect $\tilde{\mathcal{V}}$ transversely, it is the limit of the intersections of $\tilde{\mathcal{V}}$ with arbitrarily small perturbations of $\mathbf{\Phi}$ that do intersect $\tilde{\mathcal{V}}$ transversely. Let γ_n be a sequence of such transverse intersection classes converging to $\gamma := \mathtt{H} \cup \mathtt{S}$. Because $\tilde{\mathcal{V}}$ is smooth, the global product structure on a neighborhood of $\tilde{\mathcal{V}}$ from the Thom lemma implies that as d-chains,

$$\mathbf{\Phi}(\cdot, -1) - \mathbf{\Phi}(\cdot, 1) = \gamma_n \times S_1 \to \gamma \times S_1,$$

and hence that γ represents the intersection class.

Finally, we determine the orientation via a different perturbation argument. Choose a point $p \in \mathcal{S}'$, say for specificity $p = (0, \ldots, 0, i)$. The tangent space $T_p(\mathcal{S}')$ is the span of the vectors $i\mathbf{e}_k$ for $2 \leq k \leq d-1$. The tangent space $T_p(\mathtt{S})$ is obtained by adding the basis vector $\mathbf{e}_1$, while the span of the tangent space $T_p(\mathtt{H})$ is obtained by adding instead the basis vector $i\mathbf{e}_1$. We see that near $\mathcal{S}'$, γ has a product structure $\mathcal{S}' \times \mathbf{W}$, where $\mathbf{W}$ is diffeomorphic to two crossing lines, with tangent cone $xy = 0$ in the plane $\langle \mathbf{e}_1, i\mathbf{e}_1 \rangle$, as in the black lines in Figure 2.

Now perturb the homotopy as follows. Let $u : [-1, 1] \to \mathbb{R}$ be a smooth function that is equal to 1 on $[-1/4, 1/4]$ and vanishes outside of $(-1/2, 1/2)$. Define $\mathbf{\Phi}_\varepsilon(\mathbf{y}, t) := t\mathbf{e}_1 + \varepsilon u(t)\mathbf{e}_d + i\mathbf{y}$ where ε is a real number whose magnitude will be chosen sufficiently small and whose sign could be either positive or negative. Because $\mathtt{S}$ and $\mathtt{H}$ intersect only on the subset of $\mathbf{\Phi}$ where $t = 0$, their Hausdorff distance on the set $t \notin (-1/4, 1/4)$ is positive; it follows that for sufficiently small $|\varepsilon|$, the intersection of $\mathbf{\Phi}_\varepsilon$ with $\tilde{\mathcal{V}}$ is in the subset of $\mathbf{\Phi}_\varepsilon$ where $-1/4 \leq t \leq 1/4$. There $u = 1$ and the equations for the intersection γ_ε are modified from (4.2) – (4.5) as follows:

$$|x_1| \leq 1 \qquad (4.2)$$

$$x_j = 0 \;\; (2 \leq j \leq d-1), \quad x_d = \varepsilon \qquad (4.3)'$$

$$x_1^2 - y_1^2 = 1 - \varepsilon^2 - \sum_{j=2}^d y_j^2 \qquad (4.4)'$$

$$x_1 y_1 = -\varepsilon y_d \,. \qquad (4.5)'$$

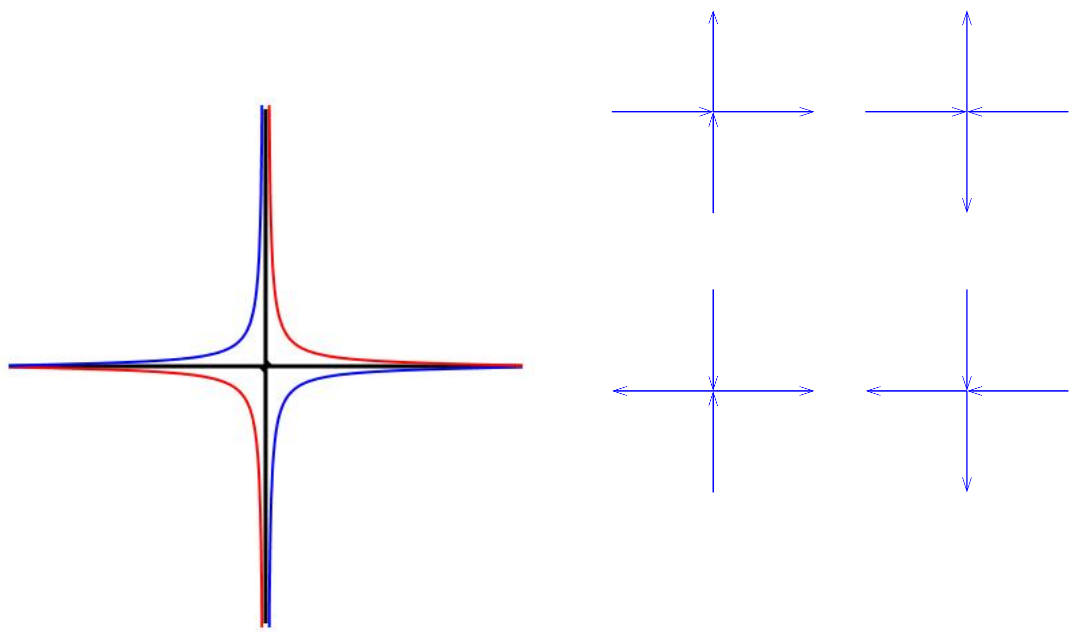

Figure 2: (left) The black line shows $\mathbf{W}$; the blue line shows the projections to the x_1-y_1 plane of $\mathbf{W}_\varepsilon$ when $y_d > 0$ and ε is small and positive; the red line shows the projections when ε is small and negative; (right) orientations of $\mathbf{W}$ consistent with the blue hyperbola.

Although $\mathbf{\Phi}_\varepsilon$ and $\tilde{\mathcal{V}}$ still do not intersect transversely, the intersection set $\gamma_\varepsilon := \mathbf{\Phi}_\varepsilon \cap \tilde{\mathcal{V}}$ is now a manifold. We now fix $y_2, \ldots, y_{d-1}$ at a value $\mathbf{y}$ inside the unit ball, setting $x_2 = \varepsilon$ and solving (4.4)' and (4.5)' for y_1 and y_d as a function of x_1. For $x_1^2 < 1 - |\mathbf{y}|^2$, as $\varepsilon \downarrow 0$, there are two components of the solution, with $y_d \to \pm\sqrt{1 - x_1^2 - |\mathbf{y}|^2}$ respectively. These correspond to different points on the sphere. Fixing one, say with $y_d > 0$, locally γ_ε has a product structure $\mathcal{S}' \times \mathbf{W}_\varepsilon$, where $\mathbf{W}_\varepsilon$ is a hyperbola in quadrants II and IV; see the blue curve in Figure 2. The (oriented) chains $\mathbf{W}_\varepsilon$ converge to $\mathbf{W}$ as $\varepsilon \downarrow 0$, therefore the possible orientations for $\mathbf{W}$ are one of the four shown on the right of Figure 2. The (oriented) chains $\mathbf{W}_\varepsilon$ also converge to $\mathbf{W}$ as $\varepsilon \uparrow 0$, narrowing the choices to the second and third choices in Figure 2, and proving the desired result. $\square$

Theorem 4.3. *Let* $\mathtt{n}$ *be the chain given by* $\mathtt{S}$ *with orientation reversed in the southern hemisphere; in other words,* $\mathtt{n}$ *is a sphere, oriented the same as the northern hemisphere of* $\mathtt{S}$. *When* d *is even, the chain* γ *is homotopic to* $\mathtt{n}$ *in* $H_{d-1}(\tilde{\mathcal{V}})$.

Proof: Let $X_1 := \mathbb{R} \times \mathcal{S}'$ and $\iota_1 : \mathcal{S}' \to X_1$ be the embedding $\mathbf{y} \mapsto (0, \mathbf{y})$. Let $X_2 = [-\pi/2, \pi/2] \times \mathcal{S}'$ and $\iota_2 : \mathcal{S}' \to X_2$ be the embedding $\mathbf{y} \mapsto (0, \mathbf{y})$. Let X denote the space obtained by gluing X_1 to X_2 modulo the identification of ι_1 and ι_2 (which conveniently identifies identically named points $(0, \mathbf{y})$ in X_1 and X_2). If for $j = 1, 2$ there are homotopies $T_j : X_j \times [0,1] \to \tilde{\mathcal{V}}$ making the maps in Figure 4 commute, then their union modulo the identification is a homotopy $T : X \times [0,1] \to \tilde{\mathcal{V}}$.

To prove the lemma, it suffices to construct these in such a way that T_2 is a homotopy from $\mathtt{S}$ to $\mathtt{n}$ and T_1 is a homotopy from $\mathtt{H}$ to a null homologous chain. On X_1, let ρ denote the $\mathbb{R}$ coordinate on X_1 and σ denote the $\mathcal{S}'$ coordinate. Let $\mathbf{z}$ denote coordinates 2 through d and let $\mathbf{x}$ and $\mathbf{y}$ denote respectively the real and imaginary parts of $\mathbf{z}$. Let t denote the $[0,1]$-coordinate of $X_1 \times [0,1]$. We may then define the homotopy T_1

$$
\begin{tikzcd}
 & X_1 \times [0,1] \arrow[dr, "T_1"] & \\
\mathcal{S}' \times [0,1] \arrow[ur, "\iota_1 \times [0,1]"] \arrow[dr, "\iota_2 \times [0,1]"'] & & \tilde{\mathcal{V}} \\
 & X_2 \times [0,1] \arrow[ur, "T_2"'] &
\end{tikzcd}
$$

Figure 3: commuting homotopies define a homotopy on the identification space X

via the equations

$$
\begin{aligned}
x_0 &= \sin(\tfrac{\pi}{2}t)\cosh(\rho);\\
y_0 &= \cos(\tfrac{\pi}{2}t)\sinh(\rho);\\
x &= \sin(\tfrac{\pi}{2}t)\sigma\sinh\rho;\\
y &= \cos(\tfrac{\pi}{2}t)\sigma\cosh\rho,
\end{aligned}
\tag{4.9}
$$

and check that $T_1((\rho,\sigma),0)$ parametrizes $\mathtt{H}$ via

$$
y_1 = \sinh(\rho), \qquad \mathbf{y} = \cosh(\rho)\sigma\,.
$$

Next we define the map $\tau : [-\pi/2,\pi/2]\times[0,1] \to [-\pi/2,\pi/2]$ by $\tau(\rho,t) = (1-t)\rho + t(\min(2\rho,0)-\pi/2)$. This is a linear homotopy from the identity to the map $\rho \mapsto \min(2\rho,0)-\pi/2$, pictured in Figure 4. Define T_2 by the equations

$$
\begin{aligned}
x_0 &= \sin(\tau(\rho,t))\ ;\\
y &= \cos(\tau(\rho,t))\sigma\ .
\end{aligned}
\tag{4.10}
$$

Again, we verify that $T_2((\rho,\sigma),0)$ parametrizes the chain $\mathtt{S}$ via the parametrization $x_0 = \sin(\rho), \mathbf{y} = \cos(\rho)\sigma$. The parametrization is not one to one, mapping the set $\{-\pi/2\}\times\mathcal{S}'$ to the south pole and $\{\pi/2\}\times\mathcal{S}'$ to the north pole; however it defines a singular chain homotopy equivalent to a standard parametrization of $\mathcal{S}'$.

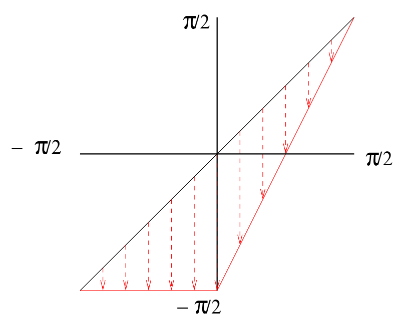

Figure 4: the linear homotopy τ

Thirdly, we check that the diagram in Figure 4 commutes, mapping $(\mathbf{y},t)$ in both cases to the point $(\sin(t\pi/2), i\cos(t\pi/2)y_2,\ldots,i\cos(t\pi/2)y_d)$. Fourthly, we check that T_2 is a homotopy from $\mathtt{S}$ to $\mathfrak{n}$. This is clear because the homotopy T_2 leaves the (generalized) longitude component alone while pushing all the southern latitudes to the south pole and stretching the northern latitudes to cover all the latitudes.

Finally, we check that T_1 is a homotopy from $\mathtt{H}$ to a null-homologous chain. The map $T_1(\cdot, 1)$ maps the imaginary hyperboloid $\mathtt{H}$ parametrized by (ρ, σ) into the $\{x_1 > 0\}$ branch of the real two sheeted hyperboloid $\mathcal{H}'$ defined by $x_1^2 = 1 + \sum_{j=2}^d x_j^2$ and parametrized by cylindrical coordinates (r, σ'). The parametrization is a double covering, with (ρ, σ) and $(-\rho, \sigma)$ getting mapped to the same point. We need to check that the orientations at (ρ, σ) and $(-\rho, -\sigma)$ are opposite. We may parametrize $\mathtt{H}$ by its projection $\mathbf{x}$ onto the last $d-1$ coordinates, then, still preserving orientation, by polar coordinates (r, σ') where $r > 0$ is the magnitude and $\sigma'(\mathbf{x}) = \mathbf{x}/r$ for $r > 0$ and anything when $r = 0$. In these coordinates, the point $(\rho, \sigma) \in \mathtt{H}$ gets mapped to the point

$$(r, \sigma') = \begin{cases} (\sinh(\rho), \sigma) & \rho > 0 \\ (-\sinh(\rho), -\sigma) & \rho < 0 \end{cases}.$$

Recalling that the orientation form on $\mathtt{H}$ is given by $\operatorname{sgn}(\rho) d\rho \wedge d\sigma$, the Jacobian is therefore given by

$$\frac{D(\sigma', r)}{D(\sigma, \rho)} = \begin{cases} \frac{d\sigma \wedge \cosh(\rho) d\rho}{d\sigma \wedge d\rho}, & \rho > 0; \\ \frac{d(-\sigma) \wedge (-\cosh(\rho)) d\rho}{-d\sigma \wedge d\rho}, & \rho < 0. \end{cases} \tag{4.11}$$

The central symmetry flips the orientation exactly on even-dimensional spheres, so that (4.11) changes signs with sign of ρ exactly when $d-2$ is even. This implies that for d even, the two branches locally covering the sheet $\{x_0^2 = |x|^2 + 1, x_0 > 0\}$ receive opposite signs and the chain $T_1(\cdot, 1)$ is homologous to zero. □

In the next section we prove perturbed versions of these results leading to identification of certain homology and cohomology classes. To pave the way, we record some further facts about the intersection of the explicit homotopy $\mathbf{\Phi}$ with the quadric.

Proposition 4.4. *There are precisely two critical points for $\tilde{h}(\mathbf{z}) := -\Re\{z_1\}$ on $\tilde{\mathcal{V}}$, namely $\pm e_1$. At the higher critical point $-\mathbf{e}_1$, the unstable manifold for the downward gradient flow on $\tilde{\mathcal{V}}$ is the sphere $\mathtt{S}$, which happens to be a subset of $\mathbf{\Phi}$, with flow lines going longitudinally from the "north pole", $-\mathbf{e}_1$, to the "south pole", $\mathbf{e}_1$. The stable manifold for the downward gradient flow at the north pole is not a subset of $\mathbf{\Phi}$; it is the upper sheet $\mathtt{H}^+$ of the two-sheeted hyperboloid forming the real part of $\mathcal{V}$, namely the set $\{\mathbf{z} \in \mathbb{R}^d : z_1 > 0$ and $z_1^2 = 1 + \sum_{j=2}^d z_j^2\}$. At the south pole $-\mathbf{e}_1$ these are reversed, with the stable manifold for downward gradient flow equal to $\mathtt{S}$ and the unstable manifold being the real surface $\mathtt{H}^- := \tilde{\mathcal{V}} \cap \mathbb{R}^d$; see Figure 5.*

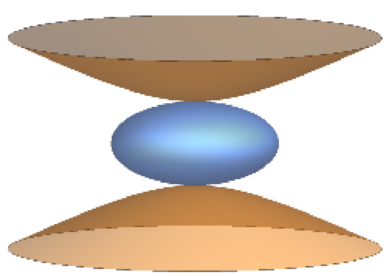

Figure 5: Stable and unstable manifolds at the critical points

Proof: Once we check that $\mathtt{S}, \mathtt{H}^+$ and $\mathtt{H}^-$ are invariant manifolds for the gradient flow on $\tilde{\mathcal{V}}$, the proposition follows from the dimensions and the fact the ranges of $\tilde{h}$ on $\mathtt{H}^+$, $\mathtt{S}$ and $\mathtt{H}^-$ respectively are $[1, \infty)$, $[-1, 1]$ and $(-\infty, -1]$. Invariance of $\mathtt{H}^\pm$ follow from the fact that the gradient is a real map (the gradient at real points is real) and therefore the real subspace, of which $\mathtt{H}^\pm$ are connected components, is preserved by gradient flow.

Invariance of $\mathtt{S}$ follows from the same argument after reparameterizing via $(x_1, \ldots, x_d) = (s_1, i\, s_2, \ldots, i\, s_d)$. $\square$

5 Perturbation of the variety

Instead of working directly with Q, we consider the small perturbations $Q_c(\mathbf{z}) := Q(\mathbf{z}) - c$. Let $\mathcal{V}_c$ denote the zero set of Q_c, $\omega_c = (P/Q_c^k)d\mathbf{z}/\mathbf{z}$ denote the corresponding d-form, and $\mathcal{M}^c = \mathbb{C}_*^d - \mathcal{V}_c$ denote the points where ω_c is analytic. Below we collect several results on the behavior of this deformation.

Proposition 5.1 (stable behavior)**.**

(i) For sufficiently small $|c| > 0$ the variety $\mathcal{V}_c$ is smooth.

(ii) For any index $\mathbf{r}$, the coefficient of the power series expansion for $F_c = P/Q_c^k$ given by (1.3),

$$a_{\mathbf{r}}(c) := \left(\frac{1}{2\pi i}\right)^d \int_T \mathbf{z}^{-\mathbf{r}} \frac{P(\mathbf{z})}{Q_c^k} \frac{d\mathbf{z}}{\mathbf{z}},$$

is holomorphic in the disk $|c| < |Q(0)|$. In particular, any given coefficient is continuous at $c = 0$ as a function of c.

Proof: The first statement is follows from the Bertini-Sard theorem (the values of c that make $\mathcal{V}_c$ singular is a finite algebraic set). The second follows from the fact that each term in the (converging, under our assumptions) expansion of

$$\frac{P}{(Q-c)^k} = \sum_{l \geq 0} \binom{-k}{l} \frac{P}{Q^{k+l}} c^l$$

is holomorphic and thus integrable over any torus in the domain of holomorphy of F, and the modulus of each term is bounded. $\square$

We will need to understand the local behavior of $h_{\mathbf{r}}$ on the smooth varieties $\mathcal{V}_c$ near $\mathbf{z}_*$. The following proposition shows that the perturbed varieties have the same geometry as the limiting quadric described in Section 4.

Proposition 5.2 (local behavior)**.** *Assume that $\hat{\mathbf{r}}$ strictly supports the tangent cone $T_{\mathbf{x}_*}(B)$. Then*

(i) There is a $\delta > 0$ such that for sufficiently small $|c| \neq 0$, there are precisely two critical points of $h_{\hat{\mathbf{r}}}$ on the variety $\mathcal{V}_c$ in the ball $B_\delta(\mathbf{z}_*)$. These points tend to $\mathbf{z}_*$ as $c \to 0$.

(ii) If c is positive and real, these critical points are real, and may be denoted $\mathbf{z}_c^{\pm}$, where

$$h_{\hat{\mathbf{r}}}(\mathbf{z}_c^+) > h_{\hat{\mathbf{r}}}(\mathbf{z}_*) > h_{\hat{\mathbf{r}}}(\mathbf{z}_c^-). \tag{5.1}$$

Proof: By part (i) of Proposition 5.1, $\mathcal{V}_c$ is smooth. The function $h_{\hat{\mathbf{r}}}$ is the real part of the logarithm of the locally holomorphic function $\mathbf{z}^{\hat{\mathbf{r}}}$ near $\mathbf{z}_*$, hence it has a critical point on the smooth complex manifold $\mathcal{V}_c$ if and only if $\mathbf{z}^{\mathbf{r}}$ does, i.e., if $d\mathbf{z}^{\mathbf{r}}$ is collinear with dQ. This latter condition defines the so-called **log-polar** variety. A local computation implies that under our conditions this is a smooth curve, intersecting $\mathcal{V}$ with multiplicity 2 at $\mathbf{z}_*$ as long as $\mathbf{r}$ is not tangent to the tangent cone $T_{\mathbf{L}(\mathbf{z}_*)}(\mathbf{z}_*)$.

Indeed, one can find a real affine-linear coordinate change such that in the new coordinates, centered at $\mathbf{z}_*$,

$$Q = z_1^2 - \sum_{k\geq 2} z_k^2 + O(|\mathbf{z}|^3) \qquad \text{and} \qquad h = z_1 + \sum_{k\geq 2} a_k z_k + O(|\mathbf{z}|^2)\,,$$

where our conditions on $\mathbf{r}$ imply $\sum_k a_k^2 < 1$. In these coordinates, the log-polar variety is given by the equation $z_k = -a_k z_0 + O(|\mathbf{z}|^2)$. Thus, the log-polar curve intersects $\mathcal{V}_c$ transversely for $|c| \neq 0$ small, and consists of 2 geometrically distinct points. A similar computation implies the second statement for real c. □

The main work in proving Theorem 2.3 will be to prove the following result.

Theorem 5.3. *Assume the hypotheses of Theorem 2.3. For $\varepsilon > 0$ and $c_* > 0$ small enough, and for any $\hat{\mathbf{r}} \in K \subseteq \widehat{\mathcal{N}}$, there is a cycle $\Gamma(\hat{\mathbf{r}})$ such that for $|c| < c_*$,*

(i) The cycle $\Gamma(\hat{\mathbf{r}})$ lies in the set $\mathcal{M}_c(-\varepsilon)$; in other words, the cycle $\Gamma(\hat{\mathbf{r}})$ lies below the height level $h_{\hat{\mathbf{r}}}(\mathbf{z}_*)-\varepsilon$ for all $\hat{\mathbf{r}} \in K$, and it avoids $\mathcal{V}_c$ for all c such that $|c| < c_*$.

(ii) There is a chain $\Gamma_c \subseteq \mathcal{M}^c$ such that $[\mathbf{T}] \simeq [\Gamma_c] + [\Gamma(\hat{\mathbf{r}})]$ in $H_d(\mathcal{M}_c)$.

(iii) The cycle $\Gamma(\hat{\mathbf{r}})$ can be chosen to be $[\mathsf{o}\gamma(\hat{\mathbf{r}})] + [\mathbf{T}(\mathbf{x}')]$, where $\gamma(\hat{\mathbf{r}})$ is a $(d-1)$ cycle in $\mathcal{V}(-\varepsilon)$, and $\mathbf{x}'$ is in a descending component B' of the complement of $\mathsf{amoeba}(Q)$ *with respect to B (see Definition 3.3).*

(iv) For fixed $\mathbf{r}$ as $c \to 0$,

$$\int_\Gamma \mathbf{z}^{-\mathbf{r}}\omega_c \quad \to \quad \int_\Gamma \mathbf{z}^{-\mathbf{r}}\omega \tag{5.2}$$

$$\int_{\Gamma_c} \mathbf{z}^{-\mathbf{r}}\omega_c \quad \to \quad 0, \text{ if } d > 2k. \tag{5.3}$$

Proof of Theorem 2.3: The first statement of Theorem 2.3 follows immediately from Theorem 5.3, as

$$a_{\mathbf{r}} = \lim_{c\downarrow 0} a_{\mathbf{r},c} = \lim_{c\downarrow 0} \int_{\mathbf{T}} \mathbf{z}^{-\mathbf{r}}\omega_c = \lim_{c\downarrow 0} \left[\int_{\Gamma(\hat{\mathbf{r}})} \mathbf{z}^{-\mathbf{r}}\omega_c + \int_{\Gamma_c} \mathbf{z}^{-\mathbf{r}}\omega_c.\right] = \int_{\Gamma(\hat{\mathbf{r}})} \mathbf{z}^{-\mathbf{r}}\omega \tag{5.4}$$

by (5.2), (5.3) and (2.3). The uniform bound in $\hat{\mathbf{r}}$ follows from compactness of K. Indeed, any cycle $\Gamma(\hat{\mathbf{r}})$ satisfies the conclusions for all $\hat{\mathbf{r}}'$ in small enough open vicinity of $\hat{\mathbf{r}}$; choosing a finite cover of K by such open vicinities, we obtain the claim. To obtain the second statement of Theorem 2.3, we use Proposition 3.5 to see that

$$\int_{\mathbf{T}(\mathbf{x}')} \mathbf{z}^{-\mathbf{r}}\omega_c \tag{5.5}$$

vanishes for all but finitely many $\mathbf{r}$. Together with (iii) of Theorem 5.3, this implies the conclusion of Theorem 2.3 for polynomial numerators. □

Theorem 5.3 is proven in Section 7.

6 Local homology near quadratic point

Recall our sign choice for Q, which implies that Q is positive on the real part of the domain of holomorphy for the Laurent expansion under consideration. We are interested in the local topology of the intersections of the singular set $\mathcal{V}_c$ with the height function $h = |\mathbf{z}^{\mathbf{r}}|$. We start with a result proved in [AGZV88, Lemma 1.3], though it dates back at least to [Mil68].

Proposition 6.1. *There exist $\delta, \delta' > 0$ such that if $B = B(\mathbf{z}_*, \delta)$ denotes the ball of radius δ about $\mathbf{z}_*$ then $\mathcal{V}_c \cap B$ is diffeomorphic to the total space of the tangent bundle to the $(d-1)$-dimensional sphere for all $c \in \mathbb{C}$ with $0 < |c| < \delta'$. In particular, the (absolute) homology groups of $\mathcal{V}_c \cap B$ are trivial in dimensions not equal to $d-1$, and $H_{d-1}(\mathcal{V}_c \cap B) \cong \mathbb{Z}$.* □

Let $h_* := h(\mathbf{z}_*)$. What we require for our results is a description of the relative homology group $H_{d-1}((\mathcal{V}_c)_* \cap B, \mathcal{V}_c \cap B(h \leq h_* - \varepsilon))$, together with explicit generators. To compute these we start with the homogeneous situation and then perturb. Denote by q the quadratic part of Q at $\mathbf{z}_*$. This is a real quadratic form, invariant with respect to conjugation, with signature $(1, d-1)$ on the real part of the tangent space at $\mathbf{z}_*$. We denote the two convex cones where $q \geq 0$ as $C_\pm$, and extend Definition 2.2 by considering supporting vectors to C_+ as well as C_-.

Consider the following three surfaces in $\mathbb{C}^d$ of respective co-dimensions 1, 1 and 2: (i) the boundary S of the unit ball; (ii) the hyperplane $H := \{\mathbf{x} + i\mathbf{y} : \mathbf{x} \cdot \hat{\mathbf{r}} = 0\}$ orthogonal to the real vector $\hat{\mathbf{r}}$; and (iii) the complex hypersurface $v := \{q = 0\}$ defined by the quadric. The transverse intersection of S and H is the equator of S.

Lemma 6.2. *If $\mathbf{r}$ is supporting, then v intersects $S \cap H$ transversely.*

Proof: By the hypothesis that $\mathbf{r}$ is supporting, one can choose h as one local coordinate, changing the rest of the coordinates so that the quadratic form q preserves its Lorentzian form. In these new coordinates it remains to prove that the functions

$$x_1 = 0, \quad x_1^2 + \sum_2^d y_k^2 - y_1^2 + \sum_2^d x_k^2 = 0, \quad x_1 y_1 - \sum_2^d x_k y_k = 0$$

have independent differentials at their common zeros outside of the origin. This is easy to check directly. □

Corollary 6.3. *For $\rho > 0$ small enough there are positive numbers ϵ_* and c_* such that the manifolds $\{\mathbf{z} : |\mathbf{z} - \mathbf{z}_*| = r\}$, $\{\mathbf{z} : h(\mathbf{z}) = h_*(\mathbf{z}) + \epsilon\}$ and $\{\mathbf{z} : Q(\mathbf{z}) = c\}$ intersect transversely, provided that $\rho/2 < r < \rho$, $\epsilon < \epsilon_*$ and $|c| < c_*$.*

Proof: For a given $\rho > 0$, introduce new coordinates in which $\mathbf{z}_*$ is the origin and the ρ-ball around $\mathbf{z}_*$ becomes the unit ball in $\mathbb{C}^d$, while rescaling Q by ρ^{-2} and h by ρ^{-1}. The resulting functions become small perturbations (decreasing with ρ) of the quadratic and linear functions in Lemma 6.2, and their zero sets become small deformations Q^ρ and H^ρ of the corresponding varieties.

In particular, the determinants whose nonvanishing witnesses the transversality of the varieties of Q^ρ, H^ρ and S are small deformations of the determinants witnessing the transversality in Lemma 6.2, and therefore are non-vanishing on some open neighborhood U of the set of solutions to $H^\rho = Q^\rho = 0$ intersected with the spherical shell where the distance to the origin is between, say, 1 and 1/2, for small enough ρ.

For small enough $\epsilon_* > 0, c_* > 0$ the sets $\{|Q^\rho| \le c_*\} \cap \{|h^\rho| \le \epsilon_*\} \cap B_1$ are contained in U. Therefore the varieties $\{Q^\rho = c\}, \{H^\rho = \epsilon\}$ and $\{|\mathbf{z}| = r\}$ are transverse when $|c| \le c_*, |\epsilon| \le \epsilon_*$ and $1/2 \le r \le 1$. $\square$

We will need one more result on the local geometry of $\mathcal{V}$ and $\{h = \text{const}\}$.

Lemma 6.4. *For $\varepsilon \ne 0$, the intersection of the real hyperplane $x_1 = -\varepsilon$ with the quadric*

$$z_1^2 - z_2^2 - \ldots - z_d^2 = c$$

is homotopy equivalent to a $(d-2)$-dimensional sphere for $|c|$ small enough.

Proof: Rescaling, we can assume that $\varepsilon = -1$. Parameterizing $(x_2, \ldots, x_d) = s\xi$ and $(y_2, \ldots, y_d) = t\eta$ where $s, t \ge 0$ and ξ, η are unit vectors in $\mathbb{R}^{d-1}$, we obtain the equations

$$x_1 = 1, \qquad 1 - y_1^2 + t^2|\eta|^2 - s^2|\xi|^2 = c, \qquad y_1 = st(\eta \cdot \xi). \tag{6.1}$$

Suppose $c = 0$. Then the manifold in question is given by

$$1 + t^2 = s^2t^2|\xi \cdot \eta|^2 + s^2.$$

Since $s^2t^2|\xi \cdot \eta|^2 + s^2 \le s^2(1 + t^2)$, one can keep ξ, η fixed and retract (s, t) satisfying this equation to $(1, 0)$. This retracts the manifold onto the unit $(d-2)$-sphere.

For nonzero c it can be verified that the manifolds given by (6.1) are transverse, and therefore remain transverse for small c, meaning the intersections are homeomorphic. $\square$

Corollary 6.5. *Assume that $\hat{\mathbf{r}}$ is supporting. Then, for $\rho > 0$ small enough, there are $\varepsilon, c_* > 0$ such that*

$$\mathcal{V}_c \cap \{h_{\hat{\mathbf{r}}} = -\varepsilon\} \cap B_\rho(\mathbf{z}_*)$$

is homotopy equivalent to $S^{(d-2)}$ for $|c| < c_*$.

Proof: We can choose coordinates in which the quadratic part of Q and $h_{\hat{\mathbf{r}}}$ are given by $z_1^2 - z_2^2 - \ldots - z_d^2$ and x_1, respectively. Then, repeating the argument in Corollary 6.3, we can view a rescaled Q and h as small perturbation of the quadratic and linear functions in the Lemma 6.4, and apply transversality. $\square$

We will be referring to the intersection

$$\mathtt{slab} := \mathtt{slab}_{\rho,\varepsilon} := B_\rho \cap \{|h - h_*| \le \epsilon\},$$

for ρ, ϵ satisfying the conditions of Corollary 6.3, as the (ρ, ϵ)-slab. We call the intersection of the slab with the boundary ∂B_ρ its **vertical** boundary, and the intersection with $h = h_* - \varepsilon$ its **bottom**.

Corollary 6.6. *For ρ, ϵ_*, c_* satisfying the conditions of Corollary 6.3, whenever $|c| < c_*$ there exists a vector field $\mathbf{v}$ on the the intersection $\mathcal{V}_{c,\mathtt{slab}} := \mathcal{V}_c \cap \mathtt{slab}_{\rho,\varepsilon_*}$ such that the following hold.*

1. *$dh \cdot \mathbf{v} < 0$ everywhere outside of the critical points of h on $\mathcal{V}_{c,\mathtt{slab}}$;*
2. *For points on $\mathcal{V}_{c,\mathtt{slab}}$ within $\rho/3$ from $\mathbf{z}_*$, the vector field is the gradient vector field for $-h$ on $\mathcal{V}_c$ with respect to the standard Hermitian form on $\mathbb{C}^d$;*

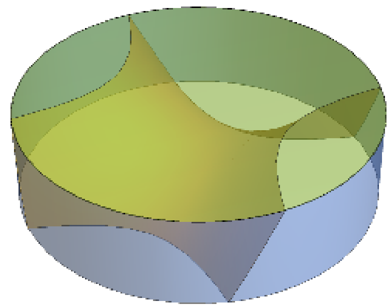

Figure 6: Slab

3. *For points at distance between $\rho/2$ and ρ from $\mathbf{z}_*$, the vector field is tangent to the spheres $\{|\mathbf{z}-\mathbf{z}_*| = \mathtt{const}\}$ and $dh \cdot \mathbf{v} = -1$;*

4. *If c is real, the vector field is invariant under conjugation: $\mathbf{v}(\overline{\mathbf{z}}) = \overline{\mathbf{v}(\mathbf{z})}$.*

PROOF: Let $\mathbf{v}^{(\mathbf{z}_*)}$ denote the gradient vector field for $-h$ on $\mathcal{V}_c$ as in 2. For any point $\mathbf{z}$ at distance between ρ and $\rho/2$ from $\mathbf{z}_*$, the transversality conclusions of the Corollary 6.3 imply that near $\mathbf{z}$ one can choose coordinates that include the four functions $h, |\mathbf{z}-\mathbf{z}_*|, \Re\{Q\}$ and $\Im\{Q\}$. In such coordinates, define $\mathbf{v}^{(\mathbf{z})} := \partial/\partial h$. Because h, Q, distance and the standard Hermitian form are invariant with respect to complex conjugation, we may choose the family $\{\mathbf{v}^{(\mathbf{z})}\}$ to be invariant, in the sense that $\mathbf{v}^{(\overline{\mathbf{z}})}(\overline{\mathbf{w}})$ is the conjugate of $\mathbf{v}^{(\mathbf{z})}(\mathbf{w})$.

Use a partition of unity to glue together the vector fields $\mathbf{v}^{(\mathbf{z})}$, ensuring that the partition gives weight 1 to $\mathbf{z}_*$ in a $\rho/3$ neighborhood of $\mathbf{z}_*$ and zero weight outside the $\rho/2$ neighborhood. This ensures conclusions 1, 2 and 3. If the partition is chosen invariant with respect to conjugation, the last conclusion will be true as well. □

Proposition 6.7. *Again, assume that $\mathbf{r}$ is supporting at $\mathbf{z}_*$, where $\mathbf{z}_*$ is a quadratic singularity of Q with signature $(1, d-1)$. Fix ρ and ϵ satisfying conditions of Corollary 6.3 and the corresponding (ρ,ε)-slab. Letting* `bottom` *denote $\mathcal{V}_c \cap \mathtt{slab} \cap \{h - h_* = -\varepsilon\}$, the relative homology group*

$$H_- := H_{d-1}(V_c \cap \mathtt{slab}, \mathtt{bottom})$$

is free of rank 2 for small enough $|c| \neq 0$. For small real $c > 0$, its generators are given by

- *an absolute cycle, the image of the generator of $H_{d-1}(V_c \cap B_r)$ under the natural homomorphism into H_-, and*
- *the relative cycle corresponding to the lobe of the real part of V_c located in $\{h \leq h_*\}$.*

PROOF: The trajectories of the flow along the vector field $\mathbf{v}(\cdot)$ constructed in Corollary 6.6 starting on $\mathcal{V}_{c,\mathtt{slab}}$ either converge to the critical points of h on $\mathcal{V}_{c,\mathtt{slab}}$, or reach `bottom`. Indeed, the value of h is strictly decreasing outside of the critical points, and cannot leave the slab through its side due to conclusion 3 of Corollary 6.6. All trajectories therefore remain in the slab or reach the bottom.

The homology of the pair $(\mathcal{V}_c \cap \texttt{slab}, \texttt{bottom})$ is generated by classes represented by the unstable manifolds of the Morse function h at critical points on $\mathcal{V}_c \cap \texttt{slab}$; this is the fundamental theorem of stratified Morse theory, for example [GM88, Theorem B]. In our situation, there are exactly two such critical points, $\mathbf{z}_-$ and $\mathbf{z}_+$, both in the real part of $\mathcal{V}_c$ and both of index $d-1$. This proves the statement about the rank of the group.

The long exact sequence of the inclusion of the bottom into $\mathcal{V}_c \cap \texttt{slab}$ gives an exact sequence containing the maps

$$H_{d-1}(\texttt{bottom}) \to H_{d-1}(\mathcal{V}_c \cap \texttt{slab}) \to H_{d-1}(\mathcal{V}_c \cap \texttt{slab}, \texttt{bottom}) .$$

Using Corollary 6.5, the first of these groups vanishes because $\mathcal{V}_c \cap \texttt{bottom}$ is homotopy equivalent to S_{d-2}. It follows that the absolute cycle generating $H_{d-1}(V_c \cap B_\rho)$ is nonvanishing in $H_{d-1}(\mathcal{V}_c \cap \texttt{slab}, \texttt{bottom})$ and is therefore a generator of H_-.

For $c > 0$, the real part of $\mathcal{V}_c$ located within the lower half of the slab, $\{h < h_*\}$, contains the critical point $\mathbf{z}_-$ (by Proposition 5.2), and the vector field $\mathbf{v}$ is tangent to it (thanks to the reality property mentioned above). Hence it coincides with the unstable manifold of $\mathbf{z}_-$. $\square$

Of course, the same argument applies to the Morse function $-h$ on V_c, implying that the group

$$H_+ := H_{d-1}(V_c \cap \texttt{slab}, (V_c \cap \texttt{slab}) \cap \{h = h_* + \epsilon\})$$

also has rank 2, and for positive real c is generated by the same absolute cycle, together with the analogous relative cycle (the lobe of the real part of $\mathcal{V}_c$ located in $\{h \geq h_*\}$). For small positive c, the situation we will restrict ourselves to from now on, we will denote the generators in H_- as $\texttt{S}_-$ and $\texttt{H}_-$, where $\texttt{S}_-$ is the absolute class represented by the small sphere in V_c and $\texttt{H}_-$ is the relative class represented by the corresponding component of the real part of Q_c. In the same way we define classes $\texttt{S}_+, \texttt{H}_+$ generating H_+.

A general duality result implies that the relative groups H_- and H_+ are dual to each other, with the coupling given by the intersection index. Briefly, the reason is that the vector field in Corollary 6.3 may be used to deform $\texttt{slab}$ until the boundary of the top flows down to the boundary of the bottom; this makes the space into a manifold with boundary satisfying the hypotheses of [Hat02, Theorem 3.43]. The conclusion of that theorem is an isomorphism between a homology group and a cohomology group, which, combined with Poincaré duality, proves the claim. In fact, we won't use this argument because we need to compute this coupling explicitly, as follows.

Proposition 6.8. *The intersection pairing between H_- and H_+ is given by*

$$\begin{aligned}
\langle \texttt{H}_+, \texttt{H}_- \rangle &= 0 ; \\
\langle \texttt{H}_+, \texttt{S}_- \rangle &= (-1)^{d(d-1)/2} ; \\
\langle \texttt{S}_+, \texttt{H}_- \rangle &= (-1)^{d(d-1)/2} ; \\
\langle \texttt{S}_+, \texttt{S}_- \rangle &= (-1)^{d(d-1)/2} \chi(S^{d-1}) = (-1)^{d(d-1)/2}(1 + (-1)^{d-1}) .
\end{aligned}$$

Remark. We pedantically distinguish between $\texttt{S}_+$ and $\texttt{S}_-$, although they are the image of the same absolute class, or even chain in V_c. Also, we note our orientations of the spheres and their tangent spaces can be in disagreement with the standard orientations induced by the complex structure. By changing the orientation of the chain $\texttt{S}$, one can suppress the annoying sign factor in the second and third equalities, but not in the last one.

PROOF: We can work (after rescaling) in the setup of Lemma 4.1. The cycles representing $\mathtt{H}_c^{\pm}$ are disjoint, explaining the first line. Each intersect $\mathtt{S}$ in precisely one point. Denoting $\partial/\partial x_k$ as ξ_k and $\partial/\partial y_k$ as η_k, the tangent spaces to $\mathtt{S} = \mathtt{S}_{\pm}$ at $\mathbf{z}_{\pm}$ are spanned by the vectors

$$\pm\eta_2, \ldots, \pm\eta_d,$$

and the tangent spaces to $\mathtt{H}_{\pm}$ at $\mathbf{z}_{\pm}$ are spanned by

$$\pm\xi_2, \ldots, \pm\xi_d,$$

respectively.

In the standard orientation of the complex hypersurface $\tilde{\mathcal{V}}$, the frame $(\xi_2, \eta_2, \ldots, \xi_d, \eta_d)$ is positive. Hence, the intersection index of $\mathtt{H}_+$ and $\mathtt{S}$ is the parity of the permutation shuffling

$$(\xi_2, \ldots, \xi_d, \eta_2, \ldots, \eta_d)$$

into that standard order, giving the second line. The third line is obtained similarly, taking the signs into account.

The last pairing can be observed by noting that the self-intersection index of a class represented by a manifold of middle dimension in a complex manifold is equal to the Euler characteristics of the conormal bundle of the manifold, under the identification of the collar neighborhood of the manifold with its conormal bundle. This gives $\chi(\mathtt{S}) = (1 + (-1)^{d-1})(-1)^{d(d-1)/2}$, where again the mismatch between the standard orientation of the conormal bundle and the ambient complex variety contributes the factor $(-1)^{d(d-1)/2}$. $\square$

Importance of the local homology computation lies in the following localization result. Let $\mathbf{u}_* := \mathbf{L}(\mathbf{z}_*) \in \mathbb{R}^d$ be a point on the boundary of $\mathsf{amoeba}(Q)$ (recall $\mathbf{L}$ is the logarithmic map $\mathbf{z} \mapsto \log|\mathbf{z}|$).

Theorem 6.9. *Assume that the quadratic critical point $\mathbf{z}_*$ is the only element of $\mathbf{T}(\mathbf{u}_*) \cap \mathcal{V}$, that $\mathbf{z}_*$ lies on the boundary of a component of $\mathsf{amoeba}(Q)^c$ and that $\mathbf{r}$ is supporting. Then for any $\rho > 0$ there exist $\varepsilon, c_* > 0$ such that for all $c \in (0, c_*)$, the intersection class $\mathcal{I}(\mathbf{T}) \subseteq \mathcal{V}_c$ can be represented by a chain supported on*

$$B_\rho(\mathbf{z}_*) \cup \{h \leq h_* - \epsilon\}.$$

PROOF: Choose ρ small enough so that the conclusions of Corollary 6.3 hold. As the intersection of $\mathcal{V}$ with the torus $\mathbf{T}(\mathbf{u}_*)$ containing $\mathbf{z}_*$ is a single point, the standard compactness arguments imply that for sufficiently small positive δ the intersection of $\mathcal{V}$ with the $\mathbf{L}$-pre-image of $B(\mathbf{u}_*, \delta)$ is contained in $B_\rho(\mathbf{z}_*)$. Pick a torus $\mathbf{T}(\mathbf{x})$ where $\mathbf{x}$ is a point in the intersection of B with the component of the complement to the amoeba defining our power series expansion. Choose $\varepsilon > 0$ such that $\{h \leq h_* - \varepsilon\}$ intersects $B_\rho(\mathbf{z}_*)$. Let $\mathbf{y}$ be a point in the component B' defined at the end of Section 3, such that $h_{\hat{\mathbf{r}}}(\mathbf{y}) < h_{\hat{\mathbf{r}}}(\mathbf{z}_*) - \varepsilon$. Choose any smooth path $\{\alpha(t) : 0 \leq t \leq 1\}$ from $\mathbf{x}$ to $\mathbf{y}$, along which $h_{\hat{\mathbf{r}}}$ decreases, and which passes through $B_\rho(\mathbf{z}_*)$. Then the $\mathbf{L}$-preimage of that path is a cobordism between $\mathbf{T}$ and a torus in $\{h \leq h(\mathbf{z}_*) - \varepsilon\}$. The transversality conclusion of Corollary 6.3 means that this cobordism, or a small perturbation of it, produces a chain realizing the intersection class $\mathcal{I}(\mathbf{T})$ and satisfying the desired conclusions. $\square$

We now come to the main result of this section, which completes Step 1 of the outline at the end of Section 3.

Theorem 6.10. *For d even the intersection class $\mathcal{I}(\mathbf{T})$ is equal to $[S_c]$ in $H_{d-1}(\mathcal{V}_c, \mathcal{V}_c(\leq -\varepsilon))$, up to sign.*

PROOF: Let e denote the class of $\mathcal{I}(\mathbf{T})$ in the relative homology group H_-. Then, by Lemma 6.7, we have $e = a\mathtt{H}_- + bS_-$ for some integers a and b. We claim that

$$\langle \mathtt{H}_+, e\rangle = \pm 1, \qquad \langle S_+, e\rangle = 0. \tag{6.2}$$

The construction of the chain representing the intersection class $\mathcal{I}(\mathbf{T})$ in Theorem 6.9 implies that it meets the chain representing $\mathtt{H}_+$ at precisely one point $\mathbf{z}'_c$. The point $\mathbf{z}'_c$ is not necessarily the point $\mathbf{z}_c^+$, but it is charactarized by being the unique point where the homotopy intersects the real variety $\mathcal{V}_{c,\mathbb{R}} \subseteq \mathcal{V}_c$.

The intersection class is represented by a chain that is smooth near $\mathbf{z}'_c$. We need to check that its intersection with the "upper lobe" $\mathtt{H}_+$ is transverse within $\mathcal{V}_c$. Indeed, one can linearly change coordinates centered at $\mathbf{z}'_c$ so that in the new coordinates $\mathbf{z}'$ the homotopy segment runs along the x'_1 axis, and thus the equations defining the cobordism are $x'_2 = \ldots = x'_d = 0$. Then $\mathcal{V}_c$ is given by $z'_1 = R(z'_2, \ldots, z'_d)$ with $dR|_{\mathbf{z}_*^+} = 0$. By direct computation, the intersection is transversal, and the tangent space to the chain representing the intersection class at $\mathbf{z}_*^+$ is the tangent space to $\mathcal{V}_c$ at $\mathbf{z}_*^+$ multiplied by i.

Because $\mathtt{H}_+$ and e intersect transversely at a single point, the first identity in (6.2) is proved. For the second identity, we again rely on perturbations of the cobordism defining the intersection class. If the path defining the cobordism avoids $\mathbf{z}_*$, for c small enough, the chain realizing $\mathcal{I}(\mathbf{T})$ constructed in Theorem 6.9 will completely avoid the chain representing $\mathtt{S}$, implying that the intersection number of e with $\mathtt{S}_+$ is zero.

To finish, we substitute (6.2) into Proposition 6.8. We compute

$$\pm 1 = \langle \mathtt{H}_+, e\rangle = a \cdot 0 + b \cdot \pm 1\,,$$

therefore $b = \pm 1$, and

$$0 = \langle \mathtt{S}_+, e\rangle = \pm a \pm b\chi(S^{d-1})\,.$$

When d is even the Euler characteristic of the $(d-1)$-dimensional sphere vanishes together with a. $\square$

7 Proof of the main theorem / Theorem 5.3

We are now ready to prove Theorem 5.3, and thus obtain our main Theorem 2.3. At each stage it is easiest to prove the result for fixed $\hat{\mathbf{r}}$ and then argue by compactness that the conclusion holds for all $\hat{\mathbf{r}} \in K$. We start with a localization result. Use the notation $\mathcal{I}_c$ to denote intersection class with respect to the perturbed variety $\mathcal{V}_c$.

Lemma 7.1. *Fix $\hat{\mathbf{r}} \in K$. Under the hypotheses of Theorem 6.10, there is an $\varepsilon > 0$ such that the intersection class $\mathcal{I}_c(\mathbf{T}, \mathbf{T}')$ is*

$$\mathcal{I}_c = [S_c] + [\gamma_c]\,,$$

where the cycle $\gamma_c(\hat{\mathbf{r}})$ representing the class $[\gamma_c] \in H_{d-1}(\mathcal{V}_c)$ is supported in $\mathcal{V}_c(< -\varepsilon)$ with respect to $h_{\hat{\mathbf{r}}}$.

PROOF: By Theorem 6.10, $\mathcal{I}_c - S_c$ is mapped to zero in the second map of the exact sequence

$$\ldots \to H_{d-1}(\mathcal{V}_c(< -\varepsilon)) \to H_{d-1}(\mathcal{V}_c) \to H_{d-1}(\mathcal{V}_c, \mathcal{V}_c(< -\varepsilon)) \to \ldots$$

Hence $\mathcal{I}_c - [S_c]$ is represented by a class in $H_{d-1}(\mathcal{V}_c(< -\varepsilon)$. $\square$

Let Σ denote the singular locus of $\mathcal{V}$, that is, the set $\{\mathbf{z} \in \mathcal{V} : \nabla Q(\mathbf{z}) = \mathbf{0}\}$. The point $\mathbf{z}_*$ is a quadratic singularity, thus isolated, and we may write $\Sigma = \{\mathbf{z}_*\} \cup \Sigma'$ where Σ' is separated from $\mathbf{z}_*$ by some positive distance.

Corollary 7.2. *If the real dimension of Σ is at most $d-2$, then for some $\delta > 0$, the cycles $\{\gamma_c(\hat{\mathbf{r}}) : 0 < |c| < \delta, \hat{\mathbf{r}} \in K\}$ may be chosen so as to be simultaneously supported by some compact Ξ disjoint from Σ.*

Remark. In the case where Σ is the singleton $\{\mathbf{z}_*\}$ or when any additional points $\mathbf{z} \in \Sigma$ satisfy $h(\mathbf{z}) \leq h(\mathbf{z}_*) - \varepsilon$ for all $\hat{\mathbf{r}} \in K$, the proof is just one line. This is all our applications presently require; however the greater generality (although most likely not best possible) may be used in future work.

Proof: The first step is to prove that for fixed $\hat{\mathbf{r}}$ we may choose $\{\gamma_c(\hat{\mathbf{r}}) : 0 < |c| < \delta\}$ satisfying the conclusion of Lemma 7.1, all supported on a fixed compact set Ξ avoiding Σ. It suffices to avoid Σ' because the condition of being supported on $\mathcal{V}_{-\varepsilon}$ immediately implies separation from $\mathbf{z}_*$. The construction in Theorem 6.9 produces a single homotopy for all c, which is then intersected with each $\mathcal{V}_c$. It follows that the union of the intersection cycles is contained in a compact set. By the dimension assumption, a small generic perturbation avoids Σ' while still being separated from $\mathbf{z}_*$.

Having seen that for fixed $\hat{\mathbf{r}}$ the cycles $\{\gamma_c(\hat{\mathbf{r}}) : 0 < |c| < \delta\}$ may be chosen to satisfy the conclusions of Lemma 7.1 and to be supported on a compact set $\Xi(\hat{\mathbf{r}})$ avoiding Σ, the rest is straightforward. For each $\hat{\mathbf{r}}$ there is a neighborhood $\mathcal{N}(\hat{\mathbf{r}}) \subseteq K$ such that $\hat{\mathbf{s}} \in \mathcal{N}$ and $h_{\hat{\mathbf{r}}}(\mathbf{z}) \leq h_{\hat{\mathbf{r}}}(\mathbf{z}_*) - \varepsilon$ imply $h_{\hat{\mathbf{s}}}(\mathbf{z}) \leq h(\mathbf{z}_*) - \varepsilon/2$. Thus we may choose $\gamma_c(\hat{\mathbf{s}}) = \gamma_c(\hat{\mathbf{r}})$ to be independent of $\hat{\mathbf{s}}$ over $\mathcal{N}(\hat{\mathbf{r}})$. Choosing a finite cover of K by these neighborhoods, the union of the corresponding sets $\Xi(\hat{\mathbf{r}})$ supports the cycles $\gamma_c(\hat{\mathbf{r}})$ for all c and $\hat{\mathbf{r}}$. $\square$

Lemma 7.3. *For any supporting $\mathbf{r}$ there exist $\varepsilon, c_* > 0$ and a compact cycle $\gamma \in H_{d-1}(\mathcal{V}(-\varepsilon))$ such that*

$$[\mathbf{T}] = [\circ S_c] + [\circ\gamma] + [\mathbf{T}'] \tag{7.1}$$

for all $|c| < c_*$.

Proof: For the compact Ξ described in Corollary 7.2, the intersection of $\mathcal{V}$ with Ξ is smooth. By Proposition 3.1, there is a neighborhood of Ξ in $\mathcal{V}_* \setminus \Sigma$ that can be parameterized as a 2-dimensional vector bundle over some compact subset $\Xi' \subseteq \mathcal{V}$. This bundle is naturally coordinatized by the values of Q so that for some small $c'_* > 0$ the tubular vicinity around $\mathcal{V}_\Xi$ can be identified with $D' \times \mathcal{V}_\Xi$ for $\mathcal{D}' := \{c \in \mathbb{C}, |c| < c'_*\}$. We will denote this vicinity as $\mathcal{V}_\Xi^{D'}$.

Lemma 7.1 implies that

$$[\mathbf{T}] = [\circ S_c] + [\circ\gamma_c] + [\mathbf{T}']$$

for all small enough $|c|$ (which we may assume from now on to be smaller than $c_* < c'_*$). The class $\circ\gamma_c$ can be represented by a small tube around a cycle $\gamma_c \in \mathcal{V}_c$, which is entirely supported by $\mathcal{V}_\Xi^{D'}$. Using the product structure $\mathcal{V}_\Xi^{D'} \cong D' \times \mathcal{V}_\Xi$ we can identify this tube with a product of a small circle (of radius $\rho(c) > 0$) around $c \in D'$ and γ_*, a cycle in the smooth part of $\mathcal{V}$ obtained by projection of γ_c. When c_* and ρ are sufficiently small, the maximum height of γ_* is $h_* - \varepsilon'$ for some $\varepsilon' > 0$.

There exists a homeomorphism of the annulus $D' - D_{\rho(c)}(c)$ fixing its outer boundary and sending the small circle $\partial D_{\rho(c)}(c)$ around c into the circle of radius c_*. Extend this homeomorphism, fiberwise, to all of the tubular vicinity $\mathcal{V}_\Xi^{D'}$. Further, extend it to the complement of $\mathcal{V}_\Xi^{D'}$ in such a way that it is identity outside of a small vicinity of $\mathcal{V}_\Xi^{D'}$ (and thus near S_c and $\mathbf{T}, \mathbf{T}'$). Choosing c_* smaller if necessary, and taking $\circ\gamma$ to be the c_*-tube around γ_* for all c with $|c| < c_*$, this cycle avoids $\mathcal{V}_c$ for all c with $|c| < c_*$ and has maximum height less than $h_* - \varepsilon$ where ε is positive once c_* has been chosen sufficiently small with respect to ε'. This completes the proof. $\square$

Proof of Theorem 5.3 Let $\gamma(\hat{\mathbf{r}})$ be chosen as in the conclusion of Lemma 7.3. Set

$$\Gamma(\hat{\mathbf{r}}) := \circ\gamma(\hat{\mathbf{r}}) + \mathbf{T}(\mathbf{x}') \,,$$

automatically satisfying condition (iii) of Theorem 5.3, and choose $\Gamma_c := \circ S_c$. Conclusion (i) follows from the choice of c_* at the end of the proof of Lemma 7.3. Conclusion (ii) is equation (7.1). As the compact cycle Γ is independent of c, equation (5.2) follows immediately from convergence of ω_c to ω on Γ for each $\mathbf{r}$. It remains only to verify (5.3).

To prove (5.3), choose a local coordinate system in which Q is reduced to its quadratic part, and rescale it by $c^{1/2}$ (either root will work). In this coordinate system $\mathbf{u} = \mathbf{v} + i\mathbf{w}$, we are integrating over the cycle $\circ S_1$, where

$$S_1 = \left\{ v_1^2 + \sum_{k=2}^{d} w_k^2 = 1; w_1 = v_2 = \ldots = v_d = 0 \right\} .$$

In the new local coordinates $\mathbf{z} = \mathbf{z}_* + c^{1/2}\mathbf{u}\psi(\mathbf{u})$ (here ψ is holomorphic, with $\psi(0) = 1$), the form $\mathbf{z}^{-\mathbf{r}}\omega_c$ becomes

$$\mathbf{z}^{-\mathbf{r}}\omega_c = \mathbf{z}_*^{-\mathbf{r}-\mathbf{1}}(1 + c^{1/2}\mathbf{u}/\mathbf{z}_*)^{-\mathbf{r}-\mathbf{1}} \frac{P(\mathbf{z}_* + c^{1/2}\mathbf{u}\psi(\mathbf{u}))}{c^k q(\mathbf{u})^k} \, c^{d/2} \, d\mathbf{u} = c^{d/2-k}\mathbf{z}_*^{-\mathbf{r}-\mathbf{1}} H(\mathbf{u}, c) \, d\mathbf{u} \,, \tag{7.2}$$

where q is the quadric (4.1) and

$$H(\mathbf{u}, c) := (1 + c^{1/2}\mathbf{u}/\mathbf{z}_*)^{-\mathbf{r}-\mathbf{1}} \frac{P(\mathbf{z}_* + c^{1/2}\mathbf{u}\psi(\mathbf{u}))}{q(\mathbf{u})^k} \,.$$

The function $H(\mathbf{u}, c)$ is holomorphic in $\mathbf{u}$ and bounded on $\circ S_1$ uniformly in c. As $c \to 0$, $H(\mathbf{u}, c) \to P(\mathbf{z}_*)/q(\mathbf{u})^k$. The conclusion (5.3) follows. $\square$

8 Application to the GRZ function with critical parameter

Having established the exponential drop, this section extends Theorem 2.3 to obtain more precise asymptotics for $a_{\mathbf{r}}$. Most of what follows concentrates on the GRZ example, however we first state a result holding more generally in the presence of a lacuna.

Theorem 8.1. *Assume the hypotheses of Theorem 2.3. Fix $\hat{\mathbf{r}}$ and let $c_1 > c_2$ be the heights of the two highest critical points, the highest being the quadric singularity. Suppose, in addition, that Q has no critical points at infinity in direction $\hat{\mathbf{r}}$ at any height in $[c_2, c_1]$. Then for every $\varepsilon > 0$ there is a neighborhood $\widehat{\mathcal{N}}$ of $\hat{\mathbf{r}}$ such that as $\mathbf{r} \to \infty$ with $\mathbf{r}/|\mathbf{r}| \in \widehat{\mathcal{N}}$,*

$$a_{\mathbf{r}} = O\left(e^{(c_2+\varepsilon)|\mathbf{r}|}\right) .$$

This is an almost immediate consequence of Theorem 2.3 and the following result:

Proposition 8.2 ([BMP19a, Theorem 2.4 *(ii)*])**.** *Let $[a, b]$ be a real interval and suppose that $\mathcal{V}_*$ has no finite or infinite critical points $\mathbf{z}$ with $h_{\hat{\mathbf{r}}}(\mathbf{z}) \in (a, b]$. Then for any $\varepsilon > 0$, any chain Γ of maximum height at most b can be homotopically deformed into a chain Γ' whose maximum height is at most $a + \varepsilon$.*

Proof of Theorem 8.1: Apply Proposition 8.2 with $a = c_2$ and $b = c_1$, resulting in the chain Γ'. Applying Theorem 2.3 and the homotopy equivalence of Γ and Γ' in $\mathcal{M}$,

$$a_{\mathbf{r}} = \int_{\Gamma'} \mathbf{z}^{-\mathbf{r}} \frac{P}{Q^k} \frac{d\mathbf{z}}{\mathbf{z}} + R$$

where R decreases super-exponentially, and in the polynomial case is in fact zero for all but finitely many $\mathbf{r}$. The height condition on Γ' implies that this integral is bounded above by the volume of Γ', multiplied by the maximum value of $|F|$ on Γ', multiplied by $e^{(c_2+\varepsilon)|\mathbf{r}|}$. $\square$

In the remainder of this section, as in Example 2.4, we let

$$F(\mathbf{z}) := \frac{1}{1 - z_1 - z_2 - z_3 - z_4 + 27 z_1 z_2 z_3 z_4} . \tag{8.1}$$

Fix $\hat{\mathbf{r}}$ to be the diagonal direction. We will prove Theorem 2.5 by first computing a very good estimate for $a_{\mathbf{r}}$, up to an unknown integer factor $\mathfrak{m}$. We then use the theory of D-finite functions and rigorous numerical bounds to find the value of $\mathfrak{m}$. Lastly, we indicate how the value of $\mathfrak{m}$ could possibly be determined by topological methods. We rely on the following more detailed topological decomposition. In order to discuss the sets $\mathcal{V}(\varepsilon)$ relative to different critical heights, we extend the notation in (2.2) via

$$\mathcal{V}_{\leq t} := \mathcal{V} \cap \{\mathbf{z} : h_{\hat{\mathbf{r}}}(\mathbf{z}) < t\} .$$

Proposition 8.3 ([BMP19a, Proposition 2.9])**.** *Let $F = P/Q$, $\mathcal{V}$, the component B and the coefficients $\{a_{\mathbf{r}}\}$ be as in Theorem 2.3. Fix $\hat{\mathbf{r}}$ and suppose the critical values are $c_1 > c_2 > \cdots > c_m$ with c_1 being the height of the quadric singularity $\mathbf{z}_*$. Suppose there are no critical points at infinity of finite height. Then there is a decomposition $\mathcal{C} = \sum_{j=1}^m \circ\gamma_i$ in $H_d(\mathcal{V}_*)$ such that for each j, $\gamma_j \in \mathcal{V}_{\leq c_j}$ and is either zero in $H_{d-1}(\mathcal{V}_{\leq c_j})$ or projects to a nonzero element of $H_{d-1}(\mathcal{V}_{\leq c_j}, H_{d-1}(\mathcal{V}_{c_j - \varepsilon}))$. The decomposition is not unique, but the least j for which $\gamma_j \neq 0$ and the projection $\pi\gamma_j$ to $H_{d-1}(\mathcal{V}_{\leq c_j}, H_{d-1}(\mathcal{V}_{c_j-\varepsilon}))$ is well defined.*

These cycles represent classes in integer homology, thus giving a representation of $a_{\mathbf{r}}$ as integer combinations of integrals over homology generators of the respective relative homology groups. Such integrals are generally computable via saddle point integration. However, determining the integer coefficients appearing in this representation can be extremely difficult, related to the so-called connection problem on solutions of differential equations. The critical points can be determined by solving the system $Q = 0$ and $\nabla Q = \lambda \nabla h_{\hat{\mathbf{r}}}$ where λ is an additional parameter, giving the following.

Proposition 8.4. *The critical points of $\mathcal{V}$ are precisely the points $\mathbf{z}_* := (1/3, 1/3, 1/3, 1/3)$, $\mathbf{w} := (\zeta, \zeta, \zeta, \zeta)$ and $\mathbf{w}' := \overline{\mathbf{w}}$, where $\zeta = (-1 + i\sqrt{2})/3$. There are no critical points at infinity. The point $\mathbf{z}_*$ is a quadric singularity.* $\square$

Let $c_1 = h_{\hat{\mathbf{r}}}(\mathbf{z}_*) = \log 81$ and $c_2 = h_{\hat{\mathbf{r}}}(\mathbf{w}) = h_{\hat{\mathbf{r}}}(\mathbf{w}') = \log 9$. Generators for the rank-2 homology group $H_3(\mathcal{V}_{\leq c_2}, \mathcal{V}_{\leq c_2 - \varepsilon})$ are given by the unstable manifold for downward gradient flows at $\mathbf{w}$ and $\mathbf{w}'$ respectively; denote these chains by γ and γ'. The conclusion of Theorem 2.3 in this case is that

$$a_{\mathbf{r}} = \int_{\Gamma} \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}}$$

where $\Gamma \in \mathcal{V}_{<c_1}$. By Proposition 8.3, $\Gamma = \mathfrak{m}o\gamma + \mathfrak{m}'o\gamma'$ in $H_3(\mathcal{V}_*)$ for some integers $\mathfrak{m}$ and $\mathfrak{m}'$, which must be equal because the coefficients are real. Explicit formulas in [PW13, Section 9.5] evaluate $\int_\gamma \mathbf{z}^{-\mathbf{r}} F d\mathbf{z}/\mathbf{z}$, which, after adding the complex conjugate, lead to the result in Theorem 2.5 with 3 replaced by $\mathfrak{m}$. To prove Theorem 2.5, it remains to determine the integer $\mathfrak{m}$.

D-finite Asymptotics and Connection Coefficients

A univariate complex function $f(z)$ is called *D-finite* if it satisfies a linear differential equation with polynomial coefficients,

$$p_r(z)f^{(r)}(z) + p_{r-1}(z)f^{(r-1)}(z) + \cdots + p_0(z)f(z) = 0\,, \tag{8.2}$$

where $p_r(z) \not\equiv 0$. We call such a linear differential equation with polynomial coefficients a **D-finite** equation. Our approach to determining $\mathfrak{m}$ relies on the fact that the diagonal of a rational function is D-finite [Chr84, Lip88], and that asymptotics of D-finite function power series coefficients can be determined up to constants which can be rigorously approximated to large accuracy. In general it is not possible to determine these constants exactly without additional information (in fact, there does not even exist a good characterization of what numbers appear as such constants) but knowing asymptotics of $a_{n,n,n,n}$ up to an *integer* allows us to immediately determine the value of $\mathfrak{m}$.

The process of determining an annihilating D-finite equation of the diagonal of a rational function lies in the domain of *creative telescoping*, a well developed area of computer algebra. In particular, there are popular packages in MAGMA [Lai16] and Mathematica [Kou10] which take a multivariate rational function and return an annihilating D-finite equation. For the running example of this section, the diagonal $f(z) = \sum_{n\geq 0} a_{n,n,n,n}z^n$ satisfies the linear differential equation

$$z^2(81z^2+14z+1)f^{(3)}(z)+3z(162z^2+21z+1)f^{(2)}(z)+(21z+1)(27z+1)f'(z)+3(27z+1)f(z) = 0. \tag{8.3}$$

The following standard results on the analysis of D-finite functions can be found in Flajolet and Sedgewick [FS09, Section VII. 9].

- The solutions of a D-finite equation form a $\mathbb{C}$-vector space, here equal to dimension three.
- A solution of (8.3) can only have a singularity when the leading polynomial coefficient $z^2(81z^2+14z+1)$ vanishes. Here the roots are 0, ζ^4, and its algebraic conjugate $\overline{\zeta}^4$, where ζ is the complex number appearing in the coordinates of the critical point c_2.
- Equation (8.3) is a *Fuchsian* differential equation, meaning its solutions have only regular singular points, and its indicial equation has rational roots. Because of this, at any point $\omega \in \mathbb{C}$, including potentially singularities, any solution of (8.3) has an expansion of the form

  $$(1-z/\omega)^\alpha \sum_{j=0}^{d} \Big(g_j(1-z/\omega)\log^j(1-z/\omega)\Big) \tag{8.4}$$

  in a disk centered at ω with a line from ω to the boundary of the disk removed, where α is rational and each g_j are analytic. At any algebraic point $z=\omega$ there are effective algorithms to determine initial terms of the expansion (8.4) for a basis of the vector space of solutions of (8.3).

- If $g(z) = \sum_{n\geq 0} c_n z^n$ is a solution of (8.3) which has no singularity in some disk $|z| < \rho$ except at a point $z = \omega$, and $g(z)$ has an expansion (8.4) in a slit disk near ω (a disk centered at ω minus a ray from the center to account for a branch cut) then asymptotics of c_n are determined by adding asymptotic contributions of the terms in (8.4). In particular, a term of the form $C(1 - z/\omega)^\alpha \log^r(1 - z/\omega)$ with $\alpha \notin \mathbb{N}$ gives an asymptotic contribution of $\omega^{-n} n^{-\alpha-1} \log^r(n) \frac{C}{\Gamma(-\alpha)}$ to c_n. Furthermore, if $g(z)$ has a finite number of singularities in a disk and each has the above form, then one can simply add the asymptotic contributions coming from each point in the disk to determine asymptotics of c_n.

These results, combined with rigorous algorithms for numerical analytic continuation of D-finite functions, allow us to rigorously determine asymptotics. For our example, the Sage ore_algebra package [KJJ15] computes a basis of solutions to (8.3) whose expansions at the origin begin

$$a_1(z) = \log(z)^2 \left(\frac{1}{2} - \frac{3z}{2} + \frac{9z^2}{2} + \cdots \right) + \log(z)\Big(-4z + 18z^2 + \cdots \Big) + \Big(8z^2 - 48z^3 + \cdots \Big)$$

$$a_2(z) = \log(z)\Big(1 - 3z + 9z^2 + \cdots \Big) + \Big(-4z + 18z^2 + \cdots \Big)$$

$$a_3(z) = 1 - 3z + 9z^2 + \cdots$$

and a basis of solutions to (8.3) whose expansions at $z = \zeta^4$ begin

$$b_1(z) = 1 + \left(\frac{13}{2} + \frac{43\sqrt{2}}{4} i \right) (z - \zeta^4)^2 + \left(\frac{8165}{48} + \frac{943\sqrt{2}}{30} i \right) (z - \zeta^4)^3 + \cdots$$

$$b_2(z) = \sqrt{z - \zeta^4} + \left(\frac{13}{3} - \frac{365\sqrt{2}}{96} i \right) (z - \zeta^4)^{3/2} - \left(\frac{7071}{1024} - \frac{1041\sqrt{2}}{32} i \right) (z - \zeta^4)^{5/2} + \cdots$$

$$b_3(z) = (z - \zeta^4) + \left(\frac{17}{3} - \frac{31\sqrt{2}}{6} i \right) (z - \zeta^4)^2 - \left(\frac{1013}{72} + \frac{1805\sqrt{2}}{36} i \right) (z - \zeta^4)^3 + \cdots .$$

Because we can compute the power series coefficients of the diagonal generating function $f(z)$ at the origin, we can represent $f(z)$ in this $a_j(z)$ basis. In fact, because $f(z)$ is analytic at the origin it must be a multiple of $a_3(z)$ and examining constant terms shows that $a_3(z) = f(z)$. Because the coefficients of $f(z)$ grow, it must admit a singularity at $z = \zeta^4$ or $z = \overline{\zeta}^4$ (in fact, we can deduce it will have a singularity at both because we already know its dominant asymptotic behaviour). If we can determine $f(z)$ in terms of the $b_j(z)$ basis then we will know its expansion in a neighbourhood of the origin, and therefore be able to determine asymptotics of its coefficients. Thus, we need to solve a *connection problem*, representing a function given by a basis specified by local information at one point in terms of a basis specified by local information at another point.

To do this it is sufficient to determine the change of basis matrix converting from the $a_j(z)$ basis into the $b_j(z)$ basis. Using algorithms going back to the Chudnovsky brothers [CC86, CC87] and van der Hoeven [vdH01], and recently improved and implemented by Mezarobba [Mez16, Mez19], we can compute this change of basis matrix numerically to any specified precision. The key is to use numeric analytic continuation to evaluate the $a_j(z)$ and $b_j(z)$ to sufficiently high precision near a fixed value of z. Such evaluations can be done using the series expansions around each point (which can be computed efficiently) and rigorous bounds on the error of series truncation [MS10].

In particular, computing the change of basis matrix in this example using the Sage implementation of Mezzarobba gives

$$f(z) = a_3(z) = C_1 b_1(z) + C_2 b_2(z) + C_3 b_3(z),$$

where C_1, C_2, and C_3 are constants which can be rigorously computed to 1000 decimal places in under 10 seconds on a modern laptop. As $b_2(z)$ is the only element of the $b_j(z)$ basis which is singular at $z = c_2$, the dominant singular term in the expansion of $f(z)$ near $z = c_2$ is

$$C_2 \sqrt{z - \zeta^4} = -\Big(\Big(3.5933098558743233\ldots\Big) + i\Big(0.38132214909311386\ldots\Big)\Big) \sqrt{z - \zeta^4}.$$

Thus, $f(z)$ has a singularity at $z = \zeta^4$ and the asymptotic contribution of this singularity to $a_{n,n,n,n}$ is

$$\Psi_1(n) := \frac{\big(4i\sqrt{2} - 7\big)^n}{n^{3/2}} \frac{\Big(\Big(0.543449606382202\ldots\Big) + i\Big(0.259547320313100\ldots\Big)\Big)}{\sqrt{\pi}} + O(9^n n^{-5/2}).$$

Repeating the same analysis at the point $z = \overline{\zeta}^4$ gives an asymptotic contribution

$$\Psi_2(n) := \frac{\big(4i\sqrt{2} + 7\big)^n}{n^{3/2}} \frac{\Big(\Big(0.543449606382202\ldots\Big) - i\Big(0.259547320313100\ldots\Big)\Big)}{\sqrt{\pi}} + O(9^n n^{-5/2}),$$

so that $a_{n,n,n,n}$ has the asymptotic expansion $a_{n,n,n,n} = \Psi_1(n) + \Psi_2(n)$.

Comparing this expansion, with numerical coefficients known to 1000 decimal places, to the expansion in (2.5) which has constants that are unknown but restricted to be integers, proves that $\lambda_2 = \lambda_3$ are integers equal to $2.99\ldots$ up to almost 1000 decimal places (almost 1000 decimal places more than needed to make this conclusion), meaning $\mathfrak{m} = 3$. This finishes the proof of Theorem 2.5. □

9 Concluding remarks

Explaining the multiplicity

We have seen that the integral over $\mathcal{C}(c_2)$ and $\mathcal{C}(c_3)$ appear in the Cauchy integral representation of $a_{n,n,n,n}$ with a multiplicity of 3. Expanding a torus past a smooth critical point leads to a coefficient of 1 when the critical point is a height maximum along the imaginary fiber and zero when it is a height minimum along this fiber. Evidently, when deforming the Cauchy domain of integration past the highest critical point $c_1 = (1/3, 1/3, 1/3, 1/3)$, the resulting chain Γ lying just below this height is not like a simple torus and instead, under gradient flow, has multiplicity 3 in the local homology basis at the diagonal points ζ and $\overline{\zeta}$.

Problem 1. *Give a direct demonstration of these coefficients being 3.*

Our best explanation at present is this. If W is a smooth algebraic hypersurface, Morse theory gives us a basis for $H_{d-1}(W)$ consisting of the unsable manifolds for downward gradient flow at each critical point. The stable manifolds at each critical point are an upper tringular dual to this via the intersection pairing. The original torus of integration is a tube over a torus T_0 in $\mathcal{V}$. If $\mathcal{V}$ were smooth, we would be trying to show that the stable manifold at $\mathbf{w}$ in $\mathcal{V}_*$ intersects T_0 with signed multiplicity ± 3, where $\mathbf{w} = (\zeta, \zeta, \zeta, \zeta)$. This is probably not true in the smooth varieties $\mathcal{V}_c$. However, as $c \to 0$, part of the stable manifold at $\mathbf{w}$

gets drawn toward $\mathbf{z}_* = (1/3, 1/3, 1/3, 1/3)$. Therefore, in the limit, we need to check how many total signed paths in the gradient field ascend from $\mathbf{w}$ to $\mathbf{z}_*$.

By the symmetry, we expect to find these paths along the three partial diagonals: $\{x = y, z = w\}$, $\{x = z, y = w\}$ and $\{x = w, y = z\}$. Solving for gradient ascents on any one of these yields three that go to $\mathbf{z}$ rather than to the coordinate planes. If these all had the same sign, the multiplicity would be 9 rather than 3, therefore, in any one partial diagonal, the three paths are two of one sign and one of the other. It remains to show that the signs are as predicted, that these are the only paths going from $\mathbf{w}$ to $\mathbf{z}_*$, and to rigorize passage from the smooth case to the limit as $c \to 0$.

Computational Morse theory

One of the central problems in ACSV is effective computation of coefficients in integer homology. Specifically, the class $[T] \in H_d(\mathcal{M})$ must be resolved as an integer combination of classes $\circ\sigma$ where $\sigma \in H_{d-1}(\mathcal{V}_*)$ projects to a homology generator for one of the attachment pairs $H_{d-1}(\mathcal{V}_{\leq c}, \mathcal{V}_{\leq c-\varepsilon})$ near a critical point with critical value c. What is known is nonconstructive. There is a highest critical value c where $[T]$ has nonzero homology in the attachment pair. The projection of $[T]$ to $H_{d-1}(\mathcal{V}_{\leq c}, \mathcal{V}_{\leq c-\varepsilon})$ is well defined. If this relative homology element is the projection of an absolute homology element $\sigma \in H_{d-1}(\mathcal{V}_{\leq c} \setminus \mathcal{V}_{\leq c-\varepsilon})$ then there is no *Stokes phenomenon*, meaning one can replace $[T]$ by $[T] - \sigma$ and continue to the next lower attachment pair where $[T] - \sigma$ projects to a nonzero homology element.

The data for this problem is algebraic. Therefore, one might hope for an algebraic solution, which can be found via computer algebra without resorting to numerical methods, rigorous or otherwise. At present, however, we have only heuristic geometric arguments.

Problem 2. *Given an integer polynomial and rational $\hat{\mathbf{r}}$, algebraically compute the highest critical points $\mathbf{z}$ for which the projection of $[T]$ to the attachment pair is nonzero. Then compute these integer coefficients. Also determine whether T is homologous to a local cycle, and in the case that it is, find a way to continue the computation to the next lower critical point.*

Combining computation and topology

One of the main achievements of the present paper is the preceding chain of reasoning that combines topological methods with computer algebra. Computer algebra methods give asymptotic formulae for the diagonal coefficients which includes an unknown constant, computable up to an arbitrarily small (rigorous) error term. These methods say nothing about the behavior of coefficients in a neighborhood of the diagonal. Topological methods show that in a neighborhood of the diagonal, coefficients are given by an asymptotic formula which is the sum of algebraic quantities up to unknown integer factors. This method on its own cannot identify the correct asymptotics without further geometric methods that have, thus far, eluded us. Combining the two analyses determines the integer factors, leading to rigorous asymptotics throughout an open cone containing the diagonal direction.

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