ECE 490: Introduction to Optimization (Homework IV)	Spring 2019

(1)

It is immediate that good will be transported from site l to k with $k < l$. So the variables are x_{kl}, the amount of goods transported from k to l, $k < l$. Altogether, $N(N-1)/2$ variables. And we need to solve

$$\begin{aligned} &\min \sum_{0\le k<l\le N} |l-k| x_{kl}, \\ &\text{s.t.} \sum_{k=0}^{l-1} x_{kl} - \sum_{k=l+1}^{N} x_{kl} = 2l - N, \qquad l = 0,\dots,N, \\ &x_{kl} \ge 0. \end{aligned}$$

Dual problem has variables $y_l, l = 0,\dots,N$,

$$\begin{aligned} &\max \sum_l y_l(2l-N), \\ &\text{s.t.} \ -y_l + y_m \le (m-l), \qquad 0 \le l < m \le N. \end{aligned}$$

It is easy to guess a primal feasible plan, say $x_{kl} = 1$ for $0 \le k < l \le N$ (by induction). The cost is

$$P = \sum_{0\le k<l\le N} (l-k) = \frac{N(N+1)(N+2)}{6}.$$

Similarly, one can guess a dual plan is $y_m = m$, and

$$\begin{aligned} D &= \sum_l l(2l-N) = 2\sum_l l^2 - N\sum_l l = 2\frac{N(N+1)(N+2)}{6} - N\frac{N(N+1)}{2} \\ &= N(N+1)\left(\frac{2N+1}{3} - \frac{N}{2}\right) = \frac{N(N+1)(N+2)}{6}. \end{aligned}$$

Note that the values of the primal and dual problems on these feasible plans are equal, $P = D$, hence, both are optimal.

(2)

We are looking at the problem of maximizing the sum of entries of the symmetric matrix

$$\begin{pmatrix} x_1 & y_1 & 0 & 0 & y_5 \\ y_1 & x_2 & y_2 & 0 & 0 \\ 0 & y_2 & x_3 & y_3 & 0 \\ 0 & 0 & y_3 & x_4 & y_4 \\ y_5 & 0 & 0 & y_4 & x_5 \end{pmatrix}$$

such that $\sum x_k = 1$ (i.e. the trace is 1) and the matrix is positive definite. The problem is convex and invariant with respect to cyclic shift of variables: $x_1 \to x_2, x_2 \to x_3, x_3 \to x_4, \dots, x_5 \to x_1$ and similarly for y. Hence, there is a minimizer with all x's and y's equal (take any minimizer, cyclic-shift variables, find the average: the result will be feasible, at least as good as the original optimum, and will have all x's and all y's equal). So, the problem reduces to maximizing $1 + 10a$, subject to

$$A(a) = \begin{pmatrix} 1/5 & a & 0 & 0 & a \\ a & 1/5 & a & 0 & 0 \\ 0 & a & 1/5 & a & 0 \\ 0 & 0 & a & 1/5 & a \\ a & 0 & 0 & a & 1/5 \end{pmatrix} \succ 0.$$

The spectrum of $A(a)$ is given by $\lambda_k = 1/5 + 2(e^{2\pi ik/5} + e^{-2\pi ik/5}) = 1/5 + 2a\cos(2\pi k/5)$. For $k = 0$, this gives $1/5 + 2a \geq 0$, for $k = 1, 4$, we have $1/5 + a\frac{-1+\sqrt{5}}{2} \geq 0$, and for $k = 2, 3$, we have $1/5 + a\frac{-1-\sqrt{5}}{2} \geq 0$. Hence, we have $a \leq \frac{2}{5(1+\sqrt{5})}$. One can easily see that for this value of a, $A(a)$ is indeed positive definite, and the sum of its coefficients is

$$1 + 10a = 1 + \frac{4}{1+\sqrt{5}} = \sqrt{5}.$$

(3)

Legendre dual is obtained as

$$\begin{aligned}
g(y_1, y_2) &= \max_{x_1,x_2} \; y_1x_1 + y_2x_2 - f(x_1, x_2) \\
&= \max_{x_1,x_2} \; y_1x_1 + y_2x_2 - \min_{s_1,s_2} \left[(x_1 - s_1)^2 + (x_2 - s_2)^2\right] \\
&= \max_{x_1,x_2,s_1,s_2} \; y_1x_1 + y_2x_2 - (x_1 - s_1)^2 - (x_2 - s_2)^2 \\
&= \max_{s_1,s_2} \Big[\Big(\max_{x_1,x_2} -y_1^2/4 + y_1(x_1 - s_1) - (x_1 - s_1)^2 - y_2^2/4 + y_2(x_2 - s_2) - (x_2 - s_2)^2\Big) \\
&\qquad\qquad + y_1^2/4 + y_2^2/4 + y_1s_1 + y_2s_2\Big] \\
&= \left[\max_{s_1,s_2} y_1s_1 + y_2s_2\right] + y_1^2/4 + y_2^2/4 = |y_1| + |y_2| + y_1^2/4 + y_2^2/4.
\end{aligned}$$

(4)

Similarly,

$$\begin{aligned}
g(y_1, y_2) &= \max_{x_1,x_2} \; y_1x_1 + y_2x_2 - f(x_1, x_2) \\
&= \max_{x_1,x_2} \; y_1x_1 + y_2x_2 - \max_{s_1,s_2} \left[(x_1 - s_1)^2 + (x_2 - s_2)^2\right] \\
&= \max_{x_1,x_2} \min_{s_1,s_2} \; \left[(y_1x_1 - (x_1 - s_1)^2) + (y_2x_2 - (x_2 - s_2)^2)\right] \\
&= \max_{x_1} \min_{s_1=\pm 1} \; (y_1x_1 - (x_1 - s_1)^2) + \max_{x_2} \min_{s_2=\pm 1} \; (y_2x_2 - (x_2 - s_2)^2).
\end{aligned}$$

Hence, we can solve the problem for each variable, x_1 and x_2 separately, i.e. we need to find g, given by

$$g(y) = \max_x yx + \min(-(x-1)^2, -(x+1)^2)$$

This is easy to do case-by-case, resulting in

$$g(y) = \begin{cases} (y/2+1)^2 - 1, & y < -2 \\ -1, & -2 \leq y \leq 2 \\ (y/2-1)^2 - 1, & \text{else} \end{cases}.$$

Using this g, the answer is $g(y_1, y_2) = g(y_1) + g(y_2)$.

(5)

For a gradient descent algorithm with step size t, the iterations are

$$\begin{aligned}
x_1(k+1) &= x_1(k) - tx_2(k) \\
x_2(k+1) &= x_2(k) - tx_1(k)
\end{aligned}$$

Thus, $x(k+1) = Ax(k)$, where $A = \begin{pmatrix} 1 & -t \\ -t & 1 \end{pmatrix}$. The eigenvalues/eigenvectors are

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \lambda_1 = 1 - t, v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \lambda_1 = 1 + t.$$

So, for an initial point, we can find a, b such that

$$x(0) = av_1 + bv_2 \implies x(100) = A^{100}x(0) = a(1-t)^{100}v_1 + b(1+t)^{100}v_2.$$

Let $t = 1/2$. Then, if $x_1(0) = x_2(0) = 1$, we have $a = 1, b = 0$, and $x_1(100) \approx 0, x_2(100) \approx 0$. Similarly, For $x_1(0) = 0.9999, x_2(0) = 1$, we have $a = 0.99995, b = 0.00005$, and $x(100)$ is a vector along $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.