(1)
It is immediate that good will be transported from site \( l \) to \( k \) with \( k < l \). So the variables are \( x_{kl} \), the amount of goods transported from \( k \) to \( l \), \( k < l \). Altogether, \( N(N - 1)/2 \) variables. And we need to solve

\[
\min \sum_{0 \leq k < l \leq N} |l - k| x_{kl},
\]

s.t.

\[
\sum_{k=0}^{l-1} x_{kl} - \sum_{k=l+1}^{N} x_{kl} = 2l - N, \quad l = 0, \ldots, N,
\]

\[x_{kl} \geq 0.\]

Dual problem has variables \( y_l, l = 0, \ldots, N, \)

\[
\max \sum_l y_l (2l - N),
\]

s.t.

\[-y_l + y_m \leq (m - l), \quad 0 \leq l < m \leq N.\]

It is easy to guess a primal feasible plan, say \( x_{kl} = 1 \) for \( 0 \leq k < l \leq N \) (by induction). The cost is

\[
P = \sum_{0 \leq k < l \leq N} (l - k) = \frac{N(N + 1)(N + 2)}{6}.
\]

Similarly, one can guess a dual plan is \( y_m = m, \) and

\[
D = \sum_l l(2l - N) = 2 \sum_l l^2 - N \sum_l l = 2 \frac{N(N + 1)(N + 2)}{6} - N \frac{N(N + 1)}{2} = N(N + 1) \left( \frac{2N + 1}{3} - \frac{N}{2} \right) = \frac{N(N + 1)(N + 2)}{6}.
\]

Note that the values of the primal and dual problems on these feasible plans are equal, \( P = D, \) hence, both are optimal.

(2)
We are looking at the problem of maximizing the sum of entries of the symmetric matrix

\[
\begin{pmatrix}
x_1 & y_1 & 0 & 0 & y_5 \\
y_1 & x_2 & y_2 & 0 & 0 \\
0 & y_2 & x_3 & y_3 & 0 \\
0 & 0 & y_3 & x_4 & y_4 \\
y_5 & 0 & 0 & y_4 & x_5
\end{pmatrix}
\]

such that \( \sum x_k = 1 \) (i.e. the trace is 1) and the matrix is positive definite. The problem is convex and invariant with respect to cyclic shift of variables: \( x_1 \rightarrow x_2, x_2 \rightarrow x_3, x_3 \rightarrow x_4, \ldots, x_5 \rightarrow x_1 \) and similarly for \( y \). Hence, there is a minimizer with all \( x \)'s and \( y \)'s equal (take any minimizer, cyclic-shift variables, find the average: the result will be feasible, at least as good as the original optimum, and will have all \( x \)'s and all \( y \)'s equal). So, the problem reduces to maximizing \( 1 + 10a \), subject to

\[
A(a) = \begin{pmatrix}
1/5 & a & 0 & 0 & a \\
a & 1/5 & a & 0 & 0 \\
0 & a & 1/5 & a & 0 \\
0 & 0 & a & 1/5 & a \\
a & 0 & 0 & a & 1/5
\end{pmatrix} \succ 0.
\]
The spectrum of $A(a)$ is given by $\lambda_k = 1/5 + 2(e^{2\pi i k/5} + e^{-2\pi i k/5}) = 1/5 + 2a \cos(2\pi k/5)$. For $k = 0$, this gives $1/5 + 2a \geq 0$, for $k = 1, 4$, we have $1/5 + a^{-1+\sqrt{5}} \geq 0$, and for $k = 2, 3$, we have $1/5 + a^{-1-\sqrt{5}} \geq 0$. Hence, we have $a \leq \frac{2}{5(1+\sqrt{5})}$. One can easily see that for this value of $a$, $A(a)$ is indeed positive definite, and the sum of its coefficients is

$$1 + 10a = 1 + \frac{4}{1+\sqrt{5}} = \sqrt{5}.$$  

(3)

Legendre dual is obtained as

$$g(y_1, y_2) = \max_{x_1, x_2} y_1 x_1 + y_2 x_2 - f(x_1, x_2)$$

$$= \max_{x_1, x_2} y_1 x_1 + y_2 x_2 - \min_{s_1, s_2} [(x_1 - s_1)^2 + (x_2 - s_2)^2]$$

$$= \max_{x_1, x_2, s_1, s_2} y_1 x_1 + y_2 x_2 - (x_1 - s_1)^2 - (x_2 - s_2)^2$$

$$= \max_{s_1, s_2} \left[ \max_{x_1, x_2} -y_1^2/4 + y_1(x_1 - s_1) - (x_1 - s_1)^2 - y_2^2/4 + y_2(x_2 - s_2) - (x_2 - s_2)^2 \right]$$

$$+ y_1^2/4 + y_2^2/4 + y_1 s_1 + y_2 s_2$$

$$= \left[ \max_{s_1, s_2} y_1 s_1 + y_2 s_2 \right] + y_1^2/4 + y_2^2/4 = |y_1| + |y_2| + y_1^2/4 + y_2^2/4.$$  

(4)

Similarly,

$$g(y_1, y_2) = \max_{x_1, x_2} y_1 x_1 + y_2 x_2 - f(x_1, x_2)$$

$$= \max_{x_1, x_2} y_1 x_1 + y_2 x_2 - \max_{s_1, s_2} [(x_1 - s_1)^2 + (x_2 - s_2)^2]$$

$$= \max_{x_1, x_2, s_1, s_2} \left[ (y_1 x_1 - (x_1 - s_1)^2) + (y_2 x_2 - (x_2 - s_2)^2) \right]$$

$$= \max_{x_1} \min_{s_1 = \pm 1} (y_1 x_1 - (x_1 - s_1)^2) + \max_{x_2} \min_{s_2 = \pm 1} (y_2 x_2 - (x_2 - s_2)^2).$$

Hence, we can solve the problem for each variable, $x_1$ and $x_2$ separately, i.e. we need to find $g$, given by

$$g(y) = \max_x y x + \min(-(x - 1)^2, -(x + 1)^2)$$

This is easy to do case-by-case, resulting in

$$g(y) = \begin{cases} (y/2 + 1)^2 - 1, & y < -2 \\ -1, & -2 \leq y \leq 2 \\ (y/2 - 1)^2 - 1, & \text{else} \end{cases}$$

Using this $g$, the answer is $g(y_1, y_2) = g(y_1) + g(y_2)$.

(5)

For a gradient descent algorithm with step size $t$, the iterations are

$$x_1(k + 1) = x_1(k) - tx_2(k)$$

$$x_2(k + 1) = x_2(k) - tx_1(k)$$

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Thus, $x(k + 1) = Ax(k)$, where $A = \begin{pmatrix} 1 & -t \\ -t & 1 \end{pmatrix}$. The eigenvalues/eigenvectors are

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \lambda_1 = 1 - t, v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \lambda_1 = 1 + t.$$

So, for an initial point, we can find $a, b$ such that

$$x(0) = av_1 + bv_2 \implies x(100) = A^{100}x(0) = a(1 - t)^{100}v_1 + b(1 + t)^{100}v_2.$$

Let $t = 1/2$. Then, if $x_1(0) = x_2(0) = 1$, we have $a = 1, b = 0$, and $x_1(100) \approx 0, x_2(100) \approx 0$. Similarly, For $x_1(0) = 0.9999, x_2(0) = 1$, we have $a = 0.99995, b = 0.00005$, and $x(100)$ is a vector along $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. 