ECE 490: Introduction to Optimization (Homework V) Spring 2019

(1)

Start with $f_k = x^{2k}, f_0 = 1, f_1 = x^2, f_2 = x^4, f_3 = x^6$. Also, note that $\int_0^\infty e^{-x}x^k dx = k!$.

$$\begin{aligned}
e_0 &= f_0 = 1, \qquad |e_0|^2 = 1 \\
e_1 &= f_1 - \frac{(e_0, f_1)}{(e_0, e_0)}e_0 = x^2 - \frac{2!}{1!}1 = x^2 - 2, \qquad |e_2|^2 = 4! - 4 \cdot 2! + 4 = 20 \\
e_2 &= f_2 - \frac{(e_0, f_2)}{(e_0, e_0)}e_0 - \frac{(e_1, f_2)}{(e_1, e_1)}e_1 = x^4 - \frac{4!}{1!} - \frac{6! - 2 \cdot 4!}{20}(x^2 - 2) = x^4 - \frac{168}{5}x^2 + \frac{216}{5} \\
e_3 &= f_3 - \frac{(e_0, f_3)}{(e_0, e_0)}e_0 - \frac{(e_1, f_3)}{(e_1, e_1)}(x^2 - 2) - \frac{(e_2, f_3)}{(e_2, e_2)}(x^4 - \frac{168}{5}x^2 + \frac{216}{5}) = \cdots
\end{aligned}$$

Computations can be done by any means...

(2)

- One can write $Q = I + \bar{e} \cdot \bar{e}^T$, where $\bar{e}^T = (1, \cdots, 1)$. So, $\sigma(Q) = 1 + \sigma(F)$, where $F = \bar{e} \cdot \bar{e}^T$. As F is rank 1 matrix, its spectrum is $(\lambda, 0, \cdots, 0)$, where λ is easy to find. As $\bar{e}$ is an eigenvector, $F\bar{e} = \bar{e} \cdot \bar{e}^T\bar{e} = \bar{e}n$. So, $\sigma(Q) = \{n + 1, 1, \cdots, 1\}$. Here, $n = 100$.

- As the problem remains the same, if one finds k-th and l-th coordiantes $(k, l > 1)$, we see that the solution is $(x, y, \cdots, y)$, and substituting we get

$$2x + (n - 1)y = 1, x + ny = 0 \implies y = -\frac{1}{n + 1}, x = \frac{n}{n + 1}.$$

- Start with $x_0 = 0$. Then, $d_0 = -(Qx - b) = e_1$ and $\alpha = \frac{|e_1|^2}{(Qe_1, e_1)} = 1/2$. It follows that

$$\begin{aligned}
x_1 &= 0 + (1/2)e_1 = (1/2, 0, \cdots, 0)^T \\
r_1 &= Qx_1 - b = (0, 1/2, \cdots, 1/2)^T \\
d_1 &= -r_1 + \frac{(Qr_1, d_0)}{(Qd_0, d_0)}d_0 = -(0, 1/2, \cdots, 1/2)^T + \left(\frac{n-1}{2}\right)\left(\frac{1}{2}\right)e_1 = \left(\frac{n-1}{4}, -1/2, \cdots, -1/2\right)^T \\
\alpha_1 &= \frac{||r_1||^2}{(Qd_1, d_1)} \\
&= \left(\frac{n-1}{4}\right) \Bigg/ \left(2\left(\frac{n-1}{4}\right)^2 + 2\left(\frac{n-1}{4}\right)(-1/2)(n-1) + (1/4)[(n-1)^2 + (n-1)]\right) \\
&= \left(\frac{n-1}{4}\right) \Bigg/ (n-1)\left(\frac{n-1}{8} + \frac{1}{4}\right) = \frac{2}{n+1} \\
x_2 &= \left(\frac{n}{n+1}, -\frac{1}{n+1}, \cdots, -\frac{1}{n+1}\right)^T.
\end{aligned}$$

(3)

- $z^3 - 9z + 8 = (z - 1)(z^2 + z - 8) = (z - 1)\left(z + \frac{1 \pm \sqrt{33}}{2}\right); z = 1, z = \frac{-1 \pm \sqrt{33}}{2}$

- Newton's method converges for any $x_0 \in (\infty, \sqrt{3}) \cup (-0.991338, 1.6439) \cup (\sqrt{3}, \infty)$. Note that $F'(\pm\sqrt{3}) = 0$.