Problem 1

(a) Yes. For example

\[ f(x) = 10 \arctan(x/10) - \arctan(5x). \]

(b) Yes, - say, the function equal to \( \sin(x)^2 \) for \( x \leq 0 \) and \( x^2 \) for \( x \geq 0 \).

|Remark: Original solution

No. If \( x_k \) are positions of maxima, and \( y_k \) are positions of minima, they cannot accumulate (if \( x_k \to x^* \leftarrow y_k \), then the function is discontinuous at \( x^* \)). So, \( x_k, y_k \to \infty \) and so the whole halfline \( \mathbb{R}_+ \) is covered by intervals \([x_k, x_{k+1}]\) where the function is between \( a \) and \( b \). So, \( \sup_{x \in \mathbb{R}} f(x) \leq a \), and the max is attained.

was for the original problem:

*Does there exist a continuous function on positive halfline \([0, \infty)\) with infinitely many local maxima and minima, such that the values at these local maxima are all equal (say, to \( a \)); the values at the local minima are all equal (say, to \( b, b < a \)) yet the function does not attain its global maximum?*

which was garbled in the test text.

Problem 2

(a) If \( x_1, x_2 \neq 0 \), the function is (locally) linear. At \( x_1 = 0 \) (or \( x_2 = 0 \)), Frechet differential does not exist.

(b) The function is the middle of 3 values, \( x_1, x_2, -x_1 - x_2 \), so it is linear whenever they are all different. The collisions happen when \( x_1 = x_2 \), or \( x_1 + 2x_2 = 0 \), or \( x_2 + 2x_1 = 0 \). For such cases the function is not differentiable.

(c) Just where \( x_1^2 - x_2^2 - 1 = 0 \), i.e., union of hyperplanes.

Problem 3

We are looking at solutions to

\[ -\cos(x) = \frac{2x}{101\pi}. \]

Every interval \((2k\pi, (2k + 2)\pi)\) has exactly two solutions to this equation for \(-25 \leq k \leq 24\), as is easy to see by inspection (\(\cos\) goes from \(-1\) to \(1\) and back to \(-1\) on these intervals, while the linear function is strictly between \(-1\) and \(1\)). This gives 50 pairs of critical points. As \( |2x/101\pi| > 1 \) for \(|x| > 101\pi/2\), it remains to check two subintervals, \([-101\pi/2, -50\pi]\), and \([50\pi, 101\pi/2]\). It is easy to see that the former interval contains one solution, and the latter none.

Altogether, 101 solutions.
Problem 4

Standard Lagrange multiplier computations gives

\[ L = \text{Tr}(XX^T + \mu X A^*) = \sum_{1 \leq k, l \leq n} (x_{kl}^2 + \mu x_{kl} a_{kl}), \]

and

\[ \frac{\partial L}{\partial x_{kl}} = 0 = 2x_{kl} + \mu a_{kl} \implies x_{kl} = -\frac{\mu}{2} a_{kl}, X = -\frac{\mu}{2} A. \]

Condition \( \text{Tr}(X A^*) = 1 \) implies \( \mu = -\frac{2}{\text{Tr}(A^* A)}, \) which gives

\[ X = \frac{1}{\text{Tr}(A^* A)} A. \]

Problem 5

Lagrange function (for \( \sum x_k \log x_k \rightarrow \min \)) is

\[ L = \sum_k x_k \log x_k - \mu_1 x_k - \mu_2 k x_k + \lambda_k x_k. \]

And,

\[ \frac{\partial L}{\partial x_k} = 0 = \log x_k + (1 - \mu_1) - \mu_2 k + \lambda_k \implies x_k = A e^{\mu_2 k}. \]

Note that \( \lambda_k = 0 \) as \( x_k > 0. \) Furthermore,

\[ A = \left( \sum_{k=0}^{\infty} e^{\mu_2 k} \right)^{-1} = 1 - e^{\mu_2}. \]

Recall that for \( q < 1, \sum_{k=0}^{\infty} k q^k = q \sum_{k=0}^{\infty} k q^{k-1} = q \left( \frac{1}{1-q} \right)' = q \frac{1}{(1-q)^2}. \]

With \( q = e^{\mu_2}, \) we conclude that

\[ \sum_{k=0}^{\infty} k x_k = (1 - e^{\mu_2}) \frac{e^{\mu_2}}{(1 - e^{\mu_2})^2} = \frac{e^{\mu_2}}{(1 - e^{\mu_2})} = a, \]

which implies that

\[ e^{\mu_2} = \frac{a}{1 + a}, \mu_2 = \log \frac{a}{1 + a} \implies x_k = \frac{a^k}{(1 + a)^{k+1}}. \]

Problem 6

Dual LP is

\[
\begin{align*}
\text{min} & \quad 5y_1 + 6y_2 + 7y_3, \\
\text{s.t.} & \quad 3y_1 + y_2 + y_3 \geq 1, \\
& \quad y_1 + 4y_2 + y_3 \geq 1, \\
& \quad y_1 + y_2 + 5y_3 \geq 1, \\
& \quad y_1, y_2, y_3 \geq 0.
\end{align*}
\]
Primal problem is solved at $x_1 = x_2 = x_3 = 1$, $z_p = 3$. So, all $y_1, y_2, y_3 > 0$, and inequalities in the dual program are equalities, and solving them yields