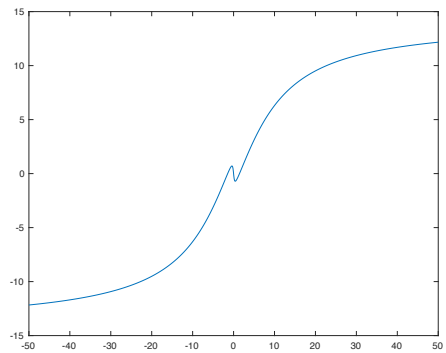
ECE 490: Introduction to Optimization (Solutions to Midterm) Spring 2019

Problem 1

(a) Yes. For example

$$f(x) = 10\arctan(x/10) - \arctan(5x).$$



(b) Yes, - say, the function equal to $\sin(x)^2$ for $x \leq 0$ and x^2 for $x \geq 0$.

[**Remark:** Original solution

No. If x_k are positions of maxima, and y_k are positions of minima, they cannot accomulate (if $x_k \to x^* \leftarrow y_k$, then the function is discontinuous at x^*). So, $x_k, y_k \to \infty$ and so the whole halfline $\mathbb{R}_+$ is covered by intervals $[x_k, x_{k+1}]$ where the function is between a and b. So, $\sup_{x\in\mathbf{R}} f(x) \leq a$, and the max is attained.

was for the original problem:

Does there exist a continuous function on positive halfline $[0, \infty)$ with infinitely many local maxima and minima, such that the values at these local maxima are all equal (say, to a); the values at the local minima are all equal (say, to $b, b < a$) yet the function does not attain its global maximum?

which was garbled in the test text.]

Problem 2

(a) If $x_1, x_2 \neq 0$, the function is (locally) linear. At $x_1 = 0$ (or $x_2 = 0$), Frechet differential does not exist.

(b) The function is the middle of 3 values, $x_1, x_2, -x_1 - x_2$, so it is linear whenever they are all different. The collisions happen when $x_1 = x_2$, or $x_1 + 2x_2 = 0$, or $x_2 + 2x_1 = 0$. For such cases the function is not differentiable.

(c) Just where $x_1^2 - x_2^2 - 1 = 0$, i.e., union of hyperplanes.

Problem 3

We are looking at solutions to

$$-\cos(x) = \frac{2x}{101\pi}.$$

Every interval $(2k\pi, (2k+2)\pi)$ has exactly two solutions to this equation for $-25 \leq k \leq 24$, as is easy to see by inspection (cos goes from -1 to 1 and back to -1 on these intervals, while the linear function is strictly between -1 and 1). This gives 50 pairs of critical points. As $|2x/101\pi| > 1$ for $|x| > 101\pi/2$, it remains to check two subintervals, $[-101\pi/2, -50\pi$, and $[50\pi, 101\pi/2]$. It is easy to see that the formar interval contains one solution, and the latter none.

Altogether, 101 solutions.

Problem 4

Standard Lagrange multiplier computations gives

$$L = \mathrm{Tr}(XX^T + \mu XA^*) = \sum_{1\le k,l\le n} \left(x_{kl}^2 + \mu x_{kl} a_{kl}\right),$$

and

$$\frac{\partial L}{\partial x_{kl}} = 0 = 2x_{kl} + \mu a_{kl} \implies x_{kl} = -\frac{\mu}{2} a_{kl}, X = -\frac{\mu}{2} A.$$

Condition $\mathrm{Tr}(XA^*) = 1$ implies $\mu = -\frac{2}{\mathrm{Tr}(AA^*)}$, which gives

$$X = \frac{1}{\mathrm{Tr}(AA^*)} A.$$

Problem 5

Lagrange function (for $\sum x_k \log x_k \to \min$) is

$$L = \sum_k x_k \log x_x - \mu_1 x_k - \mu_2 k x_k + \lambda_k x_k.$$

And,

$$\frac{\partial L}{\partial x_k} = 0 = \log x_k + (1-\mu_1) - \mu_2 k + \lambda_k \implies x_k = Ae^{\mu_2 k}.$$

Note that $\lambda_k = 0$ as $x_k > 0$. Furthermore,

$$A = \left(\sum_{k=0}^{\infty} e^{\mu_2 k}\right)^{-1} = 1 - e^{\mu_2}.$$

Recall that for

$$q < 1, \sum_{k=0}^{\infty} kq^k = q \sum_{k=0}^{\infty} kq^{k-1} = q\left(\frac{1}{1-q}\right)' = \frac{q}{(1-q)^2}.$$

With $q = e^{\mu_2}$, we conclude that

$$\sum_{k=0}^{\infty} kx^k = (1-e^{\mu_2})\frac{e^{\mu_2}}{(1-e^{\mu_2})^2} = \frac{e^{\mu_2}}{(1-e^{\mu_2})} = a,$$

which implies that

$$e^{\mu_2} = \frac{a}{1+a}, \mu_2 = \log\frac{a}{1+a} \implies x_k = \frac{a^k}{(1+a)^{k+1}}.$$

Problem 6

Dual LP is

$$\begin{aligned} \min \quad & 5y_1 + 6y_2 + 7y_3, \\ \text{s.t.} \quad & 3y_1 + y_2 + y_3 \ge 1, \\ & y_1 + 4y_2 + y_3 \ge 1, \\ & y_1 + y_2 + 5y_3 \ge 1, \\ & y_1, y_2, y_3 \ge 0. \end{aligned}$$

Primal problem is solved at $x_1 = x_2 = x_3 = 1$, $z_p = 3$. So, all $y_1, y_2, y_3 > 0$, and inequalities in the dual program are equalities, and solving them yields

$$y_1 = 6/25, y_2 = 4/25, y_3 = 3/25 \implies z_D = 3.$$