Important corollary of Gordan's lemma:

<u>Motzkin's Theorem</u>: Either the system of strict & nonstrict inequ.

$$(a_k, x) < 0 \quad k = 1 \ldots K$$
$$(b_l, x) \leq 0 \quad l = 1 \ldots L \quad \text{is consistent,}$$

Or there exist collection of weights $\{\lambda_k, k = 1 \ldots K, \mu_l, l = 1 \ldots L\}$ with all λ, μ's are nonnegative and not all of λ_k's are 0, such that

$$\sum \lambda_k \bar{a}_k + \sum \mu_l \bar{b}_l = 0.$$

[Proof in the book]

<u>Another aplication of this technique:</u>

<u>Prop</u> The following statements are equivalent:

a) Solution to system $(\bar{a}_k, x) \leq 0 \quad k = 1 \ldots K$ is just the origin, $x = 0 \in R^n$

b) Solution set to the system

$$(\bar{a}_k, x) \leq c_k \quad k = 1 \ldots K$$

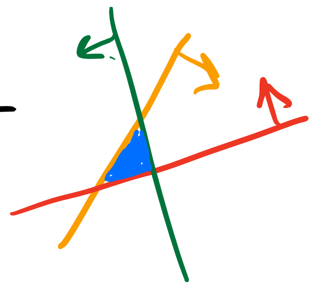
is bounded or empty for any collection of c_k

c) For some set of vectors $\bar{a}_k$, $k \in I \subset \{1, \ldots, K\}$, containing a basis of R^n, there exist <u>positive</u> λ_k:

$$\sum \lambda_k \bar{a}_k = 0$$

Corollary: any system satisfying a) contains at least $(n+1)$ vectors: $K \geq (n+1)$.

Example



Proof: a) $\Rightarrow$ b) If the set of solutions to b) is unbounded, one can find $\bar{x}_\alpha$, $\|\bar{x}_\alpha\| \to \infty$ in this set. But then $(\bar{a}_k, \bar{x}_\alpha) \le c_k \Rightarrow$

$$\Rightarrow (\bar{a}_k, \bar{x}_\alpha / \|x_\alpha\|) \le c_k / \|x_\alpha\| \to 0, \text{ and}$$

choosing a subsequence of $\bar{x}_\alpha / \|x_\alpha\| \to x_*$, we obtain that $(\bar{a}_k, \bar{x}_*) \le 0$, $\|\bar{x}_*\| = 1$ ...

b) $\Rightarrow$ c) If the system b) gives bounded solution sets, then the function

$$f(\bar{x}) = \log \sum_k e^{(a_k, x)} \text{ is coercive,}$$

and possesses a minimizer, x_*.

As in the proof of Gordan's lemma, at x_*

$$0 = \sum \lambda_k \bar{a}_k, \text{ where } \lambda_k = \frac{e^{(a_k, x_*)}}{\sum_l e^{(a_l, x_*)}} > 0.$$

So the only thing to check is whether vectors $\bar{a}_k$ are linearly independent. But were this not the case, for some vector $x_0 \neq 0$, one would have

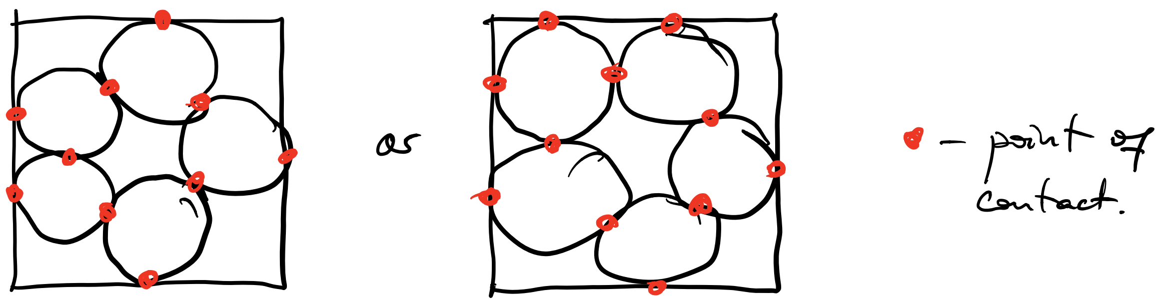
$$(\bar{a}_k, \pm\bar{x}_0) = 0, \ k = 1 \ldots k, \text{ contradicting b).}$$

c) $\Rightarrow$ a): Take $\lambda_1 \bar{a}_1 \ldots \lambda_n \bar{a}_n$ as a basis, then $\sum_{k>n} \lambda_k \bar{a}_k$ has negative coefficients.

Hence $(\bar{a}_k, x) \le 0 \Rightarrow$ coefficients of x are nonpositive and $\left(\sum_{k>n} \lambda_k a_k, x\right) \le 0$ implies that they all are 0... $\square$

Application Packing disks in a box.

Consider the problem of finding locking configuration of disks of radius r in a unit (square) box:



We say it is locking if there are no movements taking the configuration into one where disks do not touch each other or walls.

We can describe the configuration of disks by positions of their centers, so that it becomes a subset of $(\mathbb{R}^2)^n = \mathbb{R}^{2n}$ satisfying inequalities

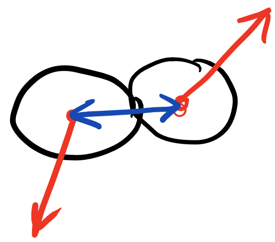
$$\left.\begin{array}{ll} \|\bar{x}_k - \bar{x}_\ell\|^2 \ge r; & 1 \le k < \ell \le n \\ r \le (\bar{x}_k, e_1) \le 1-r & 1 \le k \le n \\ r \le (\bar{x}_k, e_2) \le 1-r & \end{array}\right\}$$

n — # of disks

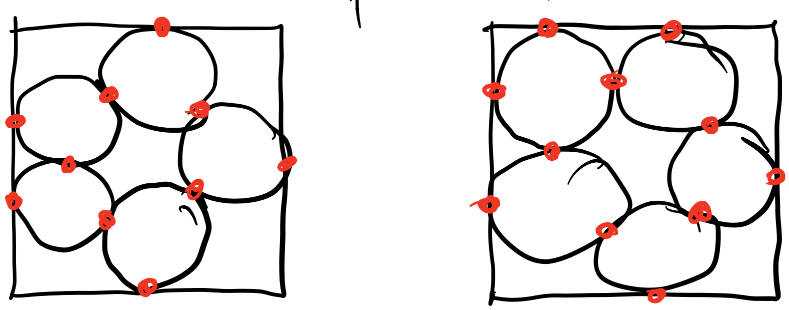
Easy to prove that existence of unlocking motion iff there is a collection of n vectors in $\mathbb{R}^2$, $\bar{V}_k \in \mathbb{R}^2$ such that

$$(\bar{V}_k - \bar{V}_\ell, \bar{x}_k - \bar{x}_\ell) \ge 0 \text{ for any disks } k, \ell \text{ contacting}$$

for any disk k touching left wall $(\bar{V}_k, \bar{e}_1) \ge 0$ etc.

(Proof? )

So, necessary condition to be locking is that the inequalities $(v_k - v_\ell, x_k - x_\ell) \geq 0$, $(v_k, e_1) \geq 0$ etc have only trivial solution, i.e. that the vectors giving these inequalities are linearly independent and a convex combination of them is $= 0$.



It follows that configuration on the left cannot be locking: it has $10 = 2 \cdot 5 = 2n$ contacts. The one on the right has 11 contacts and thus can be locking (and in fact, is).