More tools from Global Analysis

2 key results: Implicit function & Inverse function Theorems.

Implicit Function Theorem

If \( F: U \times V \to \mathbb{R}^m \), \( x_0 \in U \), \( y_0 \in V \),
\( \mathbb{R}^n \xrightarrow{F} \mathbb{R}^m \) \( F(x_0, y_0) = c \) and for
\( F_{x_0}: V \to \mathbb{R}^n \), \( F_{x_0}(y) = F(x_0, y) \), \( DF_{x_0}(y_0) \) is nonsingular (det \( \neq 0 \)).

Then for some \( r \), there is a mapping \( G: B_r(x_0) \to V \) such that
\[ F(x, G(x)) = c \] for all \( x \in B_r(x_0) \).

Example: \( F(x, y) = x^2 + y^2 \). For \( c > 0 \), \( F^{-1}(c) \) either \( \frac{\partial F}{\partial x} = 2x \), or \( \frac{\partial F}{\partial y} = 2y \neq 0 \). So, locally, circle \( S_c \) can be represented as a graph of function, either \( x = G(y) \) or \( y = G(x) \).

Remark: One often says that \( F^{-1}(c) \) is a manifold covered by patches, each of which is a smooth image of \( U \subset \mathbb{R}^n \).
Proof: First, we want to show that for a near $x_k$, there is at most one $y$ such that $F(x_k, y) = 0$. If not, there would be some $z : f_k(x_k) = z$, i.e., $f_k(x_k) = c_k = f_k(x_k)$, $k = 1, \ldots, m$. By mean value theorem, there would exist $y_k$, $0 = f_k(x_k, z) - f_k(x, y_k) = (Df_k(z, x), z - y)$, $k = 1, \ldots, m$.

But the matrix of gradients $D_y F = \left( \frac{\partial f_k}{\partial y_i} \right)$ is nonsingular at $x_k, y_k$, so (by continuity of $Df_k$) it remains nonsingular for close points $(x, y)$, $\Rightarrow$ conditions imply that $z - y = 0$, hence $y$ is unique solution near $y_k$ to $F(x, y) = 0$.

Now by Taylor formula and nonsingularity of $D_y F$, $\|F(x, y)\|^2$ for $|y| < r$, for some small $r > 0$. Hence $F(x, y) = c_k^2 > x^2$ for $|y| < r$, and $f_k(x_k, y_k) = 0$. So, $y$ is attaining minimum for each $x$ at $y$ such that $\sum (f_k - c_k)^2 > 0$ at some point $y = G(x)$. At this point, $D_y f = 0$, i.e.,

$D_y F \cdot f = 0$, and, again, using nonsingularity of $D_y F$, we conclude that $f = 0$ at $y = G(x)$. So, this defines a mapping near $x_k$ to $V$ such that $F(x, G(x)) = 0$.

One can write differential of $F$ w.r.t. $x$, arriving at

$D_x F + D_y F \cdot D_x G = 0 \Rightarrow D_x G = -D_y F \cdot D_x F$

One can see that if $F$ is $k+2$ times differentiable, then $D_x G$ is $k$ times differentiable (as $D_y F, D_x F$ are), so $G$ is in $C^k(U, V)$. \(\square\)
Corollary (Inverse function theorem) If $F: U \to \mathbb{R}^n$ is such that $DF$ is non-singular, then $F$ is (locally) invertible: there exists $G: V \to \mathbb{R}^n$ such that $G \circ F = \text{id}_{\mathbb{R}^n}$ near $0$.

Proof. Consider $H(x,y) = F(y) - x$. Then applying implicit function theorem, one gets $y = G(x)$.

Remark. Of course, this inversion is only local: $F: z \mapsto z^n \quad (|z| = 1, z \in \mathbb{C})$ maps unit circle into itself, $dF = n z^{n-1} \neq 0$, so locally $F$ is invertible, but globally each point has $n$ preimages.

Example. $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$ then $g(y) = y - a_2 y^2 + (2a_2 - a_3)y^3 + \ldots$ is a local inverse to $f$.

One can think about it in terms of trees:

\[ f = o + \underline{a} + \underline{a} + \ldots \]

\[ g = o - \underline{a} + \underline{a} - \underline{a} + \ldots \]

where $o$ is the empty tree and $\underline{a}$ is the tree with a single node labeled $a$. Each $a_i$ is a tree with two nodes labeled $a_{i-1}$ and $a_i$. The trees $o$ and $\underline{a}$ represent the constant function and the identity function, respectively.
Tangent cones & Lyusternik's Thin

**Tangent cone** describes local (1st order) behavior of a set in a linear space.

**Def.** If $X \subseteq \mathbb{R}^n$, $x \in X$, then **tangent cone** $C_x X$ (to $X$ at $x$) is the collection of vectors $V$ for some sequences $x_n \in X, \alpha_n \to 0$,

\[
V = \lim_{\alpha_n \to 0} \frac{x_n - x}{\alpha_n}
\]

(Recall that a cone is a subset of $\mathbb{R}^n$ taken into itself by dilations: $AC = C$ for any $A > 0$.)

**Example** $X = \{ x^2 - x^3 = y^2 \}$

**Lyusternik’s theorem** says that if $X = F^{-1}(c)$, $F: U \to \mathbb{R}^m$, and $DF$ maps onto $\mathbb{R}^m$ (i.e. $\text{r}_{DF} = m$),

\[
\text{then } T_x X = \text{Ker } D_x F,
\]

is an $(n-m)$-dimensional linear subspace.
Proof. Using linear coordinate changes in $\mathbb{R}^n \times \mathbb{R}^m$ we can assume that for $F = (F_1, \ldots, F_m)$, $\nabla F_k = x_k$. Then one can represent $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$, and $D_y F$ is nonsingular. Hence, $F^{-1}(c) = F^{-1}(F(a))$.

$T_x X = \{(D F)(v), v \in \mathbb{R}^{n-m}\}$, a linear subspace of $\mathbb{R}^n$, which clearly is equal to $\text{Ker } D_x F$.

Last general result from global analysis is

Morse lemma. If $x \in \text{CR}^R$ is a critical point of $f : U \rightarrow \mathbb{R}$ and $H^f(x)$ is nonsingular, then there is a coordinate change $\bar{x} = G(y)$ near $x$, such that $f(x) = f(G(y)) = f(x_0) + \sum y_k^2$, one of standard quadratic forms.

Example: $f(x_1, x_2) = x_1^2 - x_2^2 + x_1^2 x_2$, then for $x_1 = y_1 + [3], x_2 = y_2 - y_1^2 + [3]$

$f(x_1, x_2) = y_1^2 - y_2^2$ (here $[3]$ are terms of order 3 and higher).

Morse lemma is useful conceptually, but not computationally (coordinate change is rarely explicit), and we'll skip its proof.