# MORE TOOLS FROM GLOBAL ANALYSIS

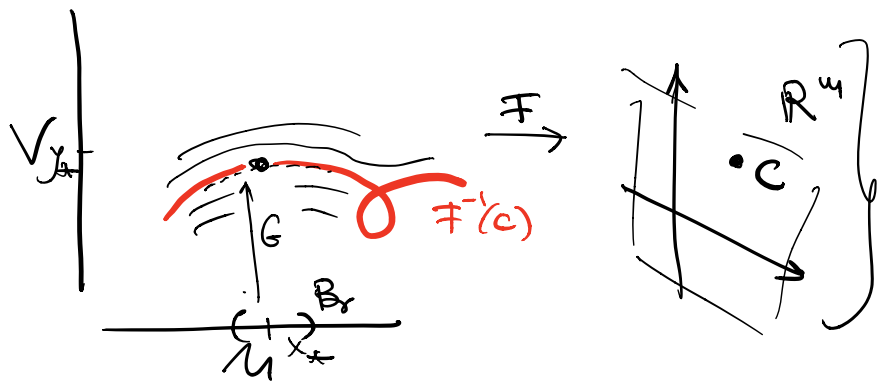
2 Key results: Implicit function & Inverse function Theorems.

**Implicit Function Theorem** $F: \underset{\mathbb{R}^n}{\mathcal{U}} \times \underset{\mathbb{R}^m}{V} \to \mathbb{R}^m$; $\mathcal{U}, V$ open; $x_* \in \mathcal{U}$, $y_* \in V$, $F(x_*, y_*) = c$ & for

$F_{x_*}: V \to \mathbb{R}^m$, $F_{x_*}(y) := F(x_*, y)$, $DF_{x_*}(y_*)$ is nonsingular ($\det \neq 0$)

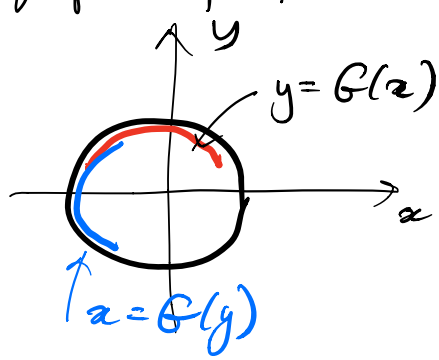
Then for some $r$, there is a smooth mapping $G: \underset{\mathcal{U}}{B_r(x_*)} \to V$ such that

$$F(x, G(x)) \equiv c \quad \text{for all } x \in B_r(x_*).$$




Preimage of a point is locally graph of a function $\mathcal{U} \to V$

**Example:** $F(x,y) = x^2 + y^2$. For $c > 0$, on $F^{-1}(c)$ either $\partial F/\partial x = 2x$, or $\partial F/\partial y = 2y \neq 0$. So, locally, circle $S_c$ can be represented as a graph of function, either $x = G(y)$ or $y = G(x)$:




Remark: One often says that $F^{-1}(c)$ is a <u>manifold</u>: covered by patches, each of which is a smooth image of $\mathcal{U} \subset \mathbb{R}^n$.

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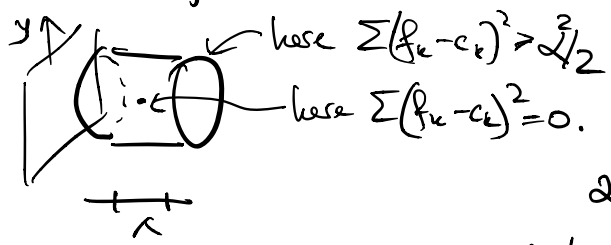
<u>Proof</u> First, we want to show that for $x$ near $x_*$, there is at most one $y$ such that $F(x,y)=c$. If not, there would be some $z: F(x,z)=c$, i.e. $f_k(x,y)=c_k=f_k(x,z)$, $k=1\dots m$. By mean value thm, there would exist $y_k$: $0=f_k(x,z)-f_k(x,y)=(\nabla_y f(x,y_k), z-y)$, $k=1\dots m$.

this is vector of partial derivatives w.r.t. $y$'s only

But the matrix of gradients $D_yF=\begin{pmatrix}\nabla_y f_1^T\\ \vdots \\ \nabla_y f_m^T\end{pmatrix}$ is nonsingular at $x_*, y_*$, so (by continuity of $\nabla f_k$) it remains nonsingular for close points $(x,y_k)$, $\Rightarrow$ conditions imply that $z-y=0$, hence $y$ is unique solution near $y_*$ to $F(x,y)=c$.

Now, by Taylor formula and nonsingularity of $D_yF$, $\|F(x_*,y)\|\geq\alpha$ for $|y|=r$, for some small $r,\alpha>0$. Hence $\psi:=\sum(f_k(x,y)-c_k)^2\geq\alpha^2$ for $\|y\|=r$, and $\psi(x_*,y_*)=0$. So, $\psi$ is attaining minimum for each $x$ at




some point $y:=G(x)$. At this point, $D_y\psi=0$, i.e. $2D_yF\cdot f=0$, and, again, using nonsingularity of $D_yF$, we conclude that $f=0$ at $y=G(x)$. So, this defines a mapping near $x_*$ to $V$ such that $F(x,G(x))\equiv c$.

One can write differential of w.r.t. $x$, arriving at

$$D_xF+D_yF\cdot D_xG=0 \Rightarrow D_xG=-D_yF^{-1}\cdot D_xF$$

$\mathbb{R}^n\to\mathbb{R}^m$ nonsingular $\mathbb{R}^m\to\mathbb{R}^m$ $\mathbb{R}^n\to\mathbb{R}^m$

One can see that if $F$ is $k+2$ times differentiable, then $D_xG$ is $(k-1)$ times differentiable (as $D_yF$, $D_xF$ are), so $G$ is in $C^k(U,V)$ $\square$

**Corollary** (Inverse function theorem) If $F: \underset{\mathbb{R}^n}{\overset{\cap}{U}} \to \mathbb{R}^n$ is such that $D_0F$ is nonsingular, then $F$ is (locally) invertible: there exists $G: \underset{\mathbb{R}^n}{\overset{\cap}{V}} \to \mathbb{R}^n$ $G \circ F = id_{\mathbb{R}^n}$ near $0$.

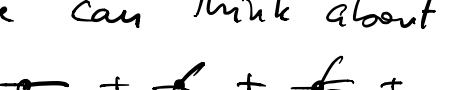
**Proof** Consider $H(x,y) = F(y) - x$. Then applying implicit function thm, one gets $y = G(x)$ $\square$

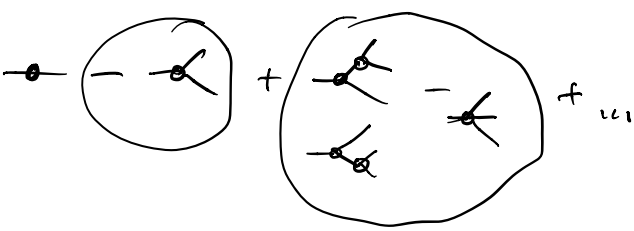
**Remark** Of course, this inversion is only local: $F: z \mapsto z^n$ $(|z|=1,\ z \in \mathbb{C})$ maps unit circle into itself; $dF = nz^{n-1} \neq 0$, so locally $F$ is invertible, but globally each point has $n$ preimages.

**Example** $f(x) = x + a_2x^2 + a_3x^3 + \dots = y$

then $g(y) = y - a_2y^2 + (2a_2^2 - a_3)y^3 + \dots$ is a local inverse to $f$

[one can think about it in terms of tree-sums

$f =$ 

$g =$ 

# TANGENT CONES & LYUSTERNIK'S THM

<u>Tangent cone</u> describes local (1st order) behavior of a set in a linear space

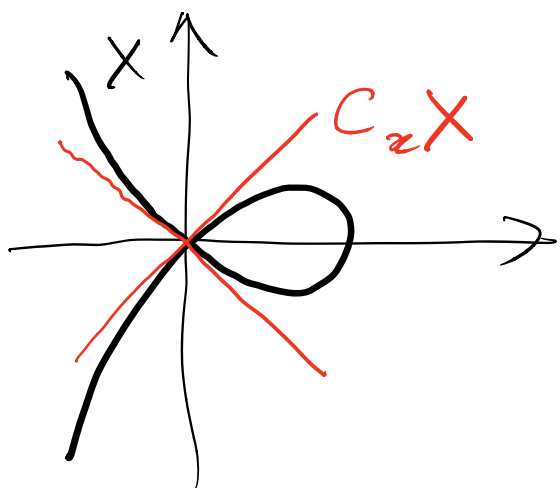
<u>Def.</u> If $X \subset \mathbb{R}^n$, $x \in X$, then <u>tangent cone</u> $C_x X$ (to $X$ at $x$) is collection of vectors $v$: for some sequences $\underset{\substack{\downarrow \\ x}}{x_n} \in X$, $\underset{\substack{\cap \\ \mathbb{R}_+}}{\alpha_n} \to 0$

$$v = \lim \frac{x_n - x}{\alpha_n}$$

$\| \; T_x(x)$ in book

(Recall that a cone is a subset of $\mathbb{R}^n$ taken into itself by dilations: $\lambda C = C$ for any $\lambda > 0$.)

<u>Example</u> $X = \{ x^2 - x^3 = y^2 \}$




Lyusternik's theorem says that if $X = F^{-1}(c)$, $F: \underset{\substack{\cap \\ \mathbb{R}^n}}{U} \to \mathbb{R}^m$, and $\underset{\substack{\cap \\ X}}{D_x F}$ maps onto $\mathbb{R}^m$ (i.e. $\operatorname{rk} D_x F = m$)

then $T_x X = \operatorname{Ker} D_x F$, is an $(n-m)$ dim linear subspace.

<u>Proof</u>. Using linear coordinate changes in $\mathbb{R}^n$ & $\mathbb{R}^m$ we can assume that for $F = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$, $\nabla f_k = x_k$. Then one can represent $\mathbb{R}^n = \underset{y}{\mathbb{R}^m} \times \underset{z}{\mathbb{R}^{n-m}}$, and $D_y F$ is nonsingular. Hence, $F^{-1}(c) = F^{-1}(F(z))$ is a graph of $G: \mathbb{R}^{n-m} \to \mathbb{R}^n$, and

$$T_x X = \{(DGv, v), v \in \mathbb{R}^{n-m}\},$$ a linear subspace of $\mathbb{R}^n$, which clearly is equal to $\operatorname{Ker} D_x F$ $\square$

Last general result from global analysis is

<u>Morse lemma</u> If $x_* \in U \subset \mathbb{R}^n$ is a critical point of $f: U \to \mathbb{R}$, and $Hf(x)$ is nonsingular, then there is a coordinate change $\bar{x} = G(\bar{y})$ near $x$, such that $f(x) = f(G(y)) = f(x_*) + \sum \pm y_k^2$, one of standard quadratic forms.

<u>Example</u>: $f(x_1, x_2) = x_1^2 - x_2^2 + x_1^2 x_2$, then for

$x_1 = y_1 + [3]$, $x_2 = y_2 - y_1^2/2 + [3]$

$f(x_1, x_2) = y_1^2 - y_2^2$ (here $[3]$ are terms of order 3 and higher).

Morse lemma is useful conceptually, but not computationally (coordinate change is rarely explicit), and we'll skip its proof