Definition: Recall that a self-adjoint operator (symmetric matrix) \( A \in \mathbb{R}^{n \times n} \) is positive semidefinite if \( v^T A v = (Av, v) = \sum_{k=1}^{n} a_{kk} v_k^2 \geq 0 \) for all \( v \in \mathbb{R}^n \).

Positive definite if \( (Av, v) > 0 \) for any \( v \neq 0 \).

Theorem: If \( f : U \to \mathbb{R} \) is twice Gateaux differentiable, and \( x \in U \) is a local minimizer, then \( Hf = \left( \frac{\partial^2 f}{\partial x_k \partial x_l} \right) \) is positive-semidefinite (and, of course, \( \nabla f = 0 \)).

Proof: Taylor formula along lines \( (x + tv) \in U \), \( v \neq 0 \). \( \square \)

Note: of course, this is just a necessary, not sufficient condition of minimality:

\( f(x) < \infty \); \( \nabla f = 0 \); \( \nabla^2 f > 0 \) (pos. semidef), but not local minimizer.

Sufficient condition:

Then \( f : U \to \mathbb{R} \); two times Frechet differentiable; \( \nabla f = 0 \); \( Hf(x) \) is positive-definite. Then \( x \) is a local minimizer. \( \square \)

Proof: Taylor formula, plus the fact that if \( Hf \) is pos. def., then for some \( c > 0 \), \( \langle Hf(v), v \rangle > 0 \). \( \square \)

Proof of: Consider the set \( \{ v \mid v \neq 0 \} \) \( \subseteq \mathbb{R}^n \) compact. Hence \( Hf(v) \) attains its minimum \( c > 0 \).

But \( \langle Hf(v), v \rangle = \langle \nabla^2 f(v), v \rangle \); \( v \neq 0 \).

Another easy but useful result. If \( f \) is positive semidefinite on open convex \( U \), and \( \nabla f(x) = 0 \), then \( x \) is a global minimizer on \( U \). Indeed, \( f(y) = f(x) + (\nabla f(x), y-x) + \frac{1}{2} \langle \nabla^2 f(x), y-x \rangle = 0 \) for some \( y \in \mathbb{R}^n \).

Note: if \( Hf \) is indefinite, then a critical pt \( x \) is a saddle.
**Definition** Quadratic form given by a matrix $A$ is called **nondegenerate** if $\det A \neq 0$.

A critical pt $x$ where $Hf$ is nondegenerate is called **Morse**.

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**Examples**

$f = 3xy - x^2 - y^2$

$\nabla f = (3y - 2xy - y^2, 3x - x^2 - 2xy)$

Critical points:

$(0,0), (1,1), (0,3), (3,0)$

$saddle$:

$H = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$

local maximum:

$\nabla^2 f = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

$\bigtriangledown H_v, v = -2v_x^2 - 2v_y v_x - 2v_y^2$

$= -v_x^2 - (v_x v_y)^2 - v_y^2 < 0$

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Systematic understanding of the structure of functions near critical pts requires analysis of quadratic forms.

$q(v) = (Hv, v) = \sum a_{\ell \ell} v_{\ell} v_{\ell}$

Easiest quadratic forms: diagonal ones, where

$H = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ \vdots & \vdots \\ 0 & \lambda_n \end{bmatrix}$, so that $q(v) = \sum \lambda_{\ell} v_{\ell}^2$

$NB: \{ \lambda_1 \leq \ldots \leq \lambda_n \}$

Spectrum of $H$.

Clearly, if all $\lambda_k > 0 (> 0)$, $k = 1, \ldots, n$ then $q$ is positive semidefinite (pos. definite). If some $\lambda_k < 0 < \lambda_1$, then $q$ is a saddle. All $\leq 0 (< 0)$ – negative semidefinite (det.)
Remarkably, any symmetric matrix can be turned into diagonal one by a change of coordinates...

\[ \begin{aligned}
\text{e.g. } & 2x^2 - 2xy + 2y^2 = 3 \left( \frac{x+y}{\sqrt{2}} \right)^2 + \left( \frac{2-y}{\sqrt{2}} \right)^2 \quad \text{remark that}
\vec{\text{vectors } (x_2, y_2), (x_2, -y_2) \text{ form an}} \\
\text{orthogonal basis} \end{aligned} \]

Namely, if \( q = \sum q_k x_k x_k \), then there is a unitary matrix \( U \) \[
\text{unitary matrices are characterized by } U^T U = E \text{, or, equivalently, by the property that their rows - as columns - for an orthogonal basis}
\] such that for coordinates defined by \( x = U y \), the resulting form \( \tilde{q}(y) = q(U y) = (A U y, U y) = U^T A U = \Sigma \) is diagonal.

The eigenvalues of a matrix do not change under similarities \( X \Rightarrow U^{-1} X U \), so the spectra of \( A \) and \( U^T A U = U^T A U \) are the same.

\[ \Omega = \{ \lambda_1, \lambda_2, \ldots, \lambda_n \} \subseteq \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \]

Recall, by the way, that spectra of symmetric matrices are real.

Last useful fact: given 2 symmetric matrices, \( A \& B \), such that \( A \) is positive definite, there is a coordinate transformation \( x = X y \), such that both \( A \& B \) become diagonal \( \Sigma_1 = X^T A X \), \( \Sigma_2 = X^T B X \).

\text{Proof: Descanso (A, x, x) new Euclidean norm. orthogonal}

In more details: One can read \( U : u^T A u = \begin{bmatrix} \lambda_k & 0 & \cdots & 0 \\ 0 & \lambda_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{bmatrix} \quad A_k > 0 \quad k = 1, \ldots, n \)

Then \( C = U \begin{bmatrix} \lambda_k & 0 & \cdots & 0 \\ 0 & \lambda_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{bmatrix} U^T \) one has: \( C^T A C = E \). Then \( C^T B C = \Rightarrow \)

is symmetric (as \( C^T = C \)) and can be represented as \( V \Sigma_2 V^T \) for some orthogonal \( V \). Set \( X = C^T V \). Then \( X^T B X = V^T C^T B C V = \Sigma_2 \), \( X^T A X = V^T C^T A C V = E \) ...
Sylvester's Criterion

How to determine whether a symmetric matrix $A$ generates a positive-definite quadratic form $q(v) = (Av, v)$?

$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

Determinants of $\square$ are called (principal) minors.

Thus, $(Av, v)$ is positive-definite iff all nested principal minors are $> 0$.

**Proof:**

$\Rightarrow$ is immediate: if $(Av, v) > 0$ for all $v$, it is also an subspace $\langle v_1, \ldots, v_k, 0, 0, \ldots \rangle$. So quadratic form on $\mathbb{R}^k$ given by $A_k = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}$ is positive definite, and so all $\lambda \in \mathbb{C}(A_k)$ are $> 0$.

So det $A_k > 0$

$\Leftarrow$. Assume it's true in $A_{n-1}$, prove for $n+1$. (Base, $n=1$, is trivial.)

$A = \begin{pmatrix} B & b \\ b^T & c \end{pmatrix}$

$B \in \text{Mat}(n \times n), \ b \in \text{Mat}(n \times 1)$ ...

If $A$ is not pos. def., there is $\lambda_{n+1} < 0$. But as (by assumption) det $A > 0$, there is $\lambda_n < 0$. Let $A u = \lambda_n u, A v = \lambda_{n+1} v$.

Now, some linear combination $w = \alpha u + \beta v$ has last coordinate $= 0$ (but $w \neq 0$). Then $(Aw, w) = (\alpha^2 \lambda_n u + \beta^2 \lambda_{n+1} v, \alpha u + \beta v)^T$

$= \lambda_n \alpha^2 (u, u) + \lambda_{n+1} \beta^2 (v, v) < 0$.

But that means that $B$ is not positive definite, contradiction.

**Example**

\[
\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}
\]

is positive definite.