0. Basic results on optimizers

\[ f : U \to \mathbb{R}, \quad U \subset \mathbb{R}^n \]

Consider \( x^* \in U \), \( B_r(x^*) \) — ball of radius \( r \) around \( x^* \). We say that

- \( x^* \) is a local minimizer if for some \( r > 0 \), \( f(x^*) \leq f(x) \), \( x \in B_r(x^*) \subset U \); strict local minimizer, \( f(x^*) < f(x) \) for all \( x \neq x^* \).
- global minimizer if same true for \( U \).
- critical point if Gateaux diff. \( \nabla f \) at \( x^* \) is \( 0 \).
- saddle point if it's a critical pt \& there are \( p, q \) pts \( x^* \), \( x^* \)
  \[ f(x^*) < f(x') \leq f(x^*) \]
  in any \( B_r(x^*) \).

0. Weierstrass' Theorem: if \( f \) is continuous on a compact set \( K \).

Then \( f \) possesses a global minimizer (or maximizer).

\[ \text{Proof: If a minimizer doesn't exist, we can find seq of points } \{ x_k \} \subset U, \quad f(x_k) \to \inf f. \]

Put, by compactness of \( K \), there is a subsequence \( \{ x_{k_i} \} \). 
\[ x_{k_i} \to x^* \in K. \]
Then \( f(x^*) = \inf f(x), \) for all \( x \in K. \)

**Reminder**

Compact sets:

- **Main definition:** open cover admits finite subcover.
  \[ \Rightarrow \text{Any sequence has convergent subsequence.} \]
- **Bounded & closed \( \subset \mathbb{R}^n \Rightarrow \text{compact} \)

A function on metric space \( D \) is called coercive if
\[ f \to \infty \quad \text{as } |x| \to \infty \]
(\( |x| = \text{dist}(x, \mathbf{0}) \)) origin in \( \mathbb{R}^n \), some basepoint respected.

**Then** if \( f \) is continuous coercive on closed subset of \( \mathbb{R}^n \),
then \( f \) has a global minimum.

**Proof:** \( f \leq c \) is either empty or bounded & closed. 
\[ \Rightarrow \text{compact.} \Rightarrow \text{has a minimum.} \]
Application

Any polynomial \( p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0 \) has a root in \( \mathbb{C} \).

Proof: If \( \|f\| \) is coercive, so it attains a global min at \( x^* \) (assume \( f(x^*) = 0 \)). But if \( p = a_0 + a_1 x + a_2 x^2 + \ldots \) then one can find \( v \) so \( a_k v^k = a_0 \), \( a_0 < 0 \), real.

For small \( v \), \( a_k v^k \) is much bigger than rest of terms \( \implies |a_0 + a_1 x + \ldots| < |a_0| \).

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2. First order optimality conditions

Neither of all optimality conditions:

Thus, if \( f: \mathbb{R} \to \mathbb{R} \) is Gateaux differentiable on open \( A \), and \( x^* \) is a minimizer, then \( \nabla f(x^*) = 0 \).

Proof: Taylor formula.

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Some examples

1. Steiner point: find point on the (base) perimeter \( AB \) such that distances to vertices of a triangle \( ABC \) minimize: \( f(0) = |AB| + |BC| + |CA| \)

   Solution: \( \nabla f = n_1 + n_2 + n_3 \Rightarrow \text{"120° rule": at optimum, } \frac{n_1}{h_2} = \frac{n_2}{h_3} = \frac{n_3}{h_1} = \frac{1}{\sqrt{3}} \).

   Caveat: what about this: \( A \) and \( C \)?

2. Euclid’s optimization problem: find \( O \) maximizing \( \text{area of parallelogram} \) optimal when \( O \) is midpoint between \( B \) and \( C \).
\[ f(x_1, \ldots, x_n) = \frac{1}{2} \sum \frac{\partial^2 f}{\partial x_k^2} - \sum_{k \neq l} \frac{1}{x_k - x_l} \]

Confining potential

\[ f \rightarrow \text{un} \]

[Background: spectrum density for Gaussian Unitary Ensemble: random Hermite matrix (remind!) of size \( n \times n \)]

\[ \frac{\partial^2 f}{\partial x_k^2} = \left( \frac{\partial}{\partial x_k} \left( \sum_{l \neq k} \frac{1}{x_k - x_l} \right) \right) \]

Take \( p = p_n = \prod_{k=1}^{n} (x - x_k) \) \( p_1 (x_k) = \prod_{l \neq k} (x_k - x_l) \)

\[ \frac{p''}{p'} (x_k) = \frac{\sum_{l \neq k} \prod_{m \neq l, m \neq k} (x_k - x_m)}{\prod_{l \neq k} (x_k - x_l)} = 2 \sum_{l \neq k} \frac{1}{x_k - x_l} \]

So, if \( (x_n) \) is unique, \( \frac{p''}{p'} (x_k) = 2x_k \)

\[ p''(x_k) - 2x_k p'(x_k) = 0 \quad \text{for all} \quad x_k \quad \text{roots of} \quad p' \]

\[ \Rightarrow \quad p_n''(x) - 2x p_n'(x) = -2n p_n(x) \]

NB: \( \int p_n^2 e^{-x^2} \, dx = 2^n \frac{\sqrt{\pi}}{2} p_n(2) \)

\[ = -2 \int \frac{p_n^2 - p_n''}{2n} e^{-x^2} \, dx = \left( \frac{p_n^2 e^{-x^2}}{2n} \right)' = (p_n' e^{-x^2})' \]

\[ = -\frac{1}{2n} \int \left( p_n^2 e^{-x^2} \right)' p_n \, dx = -\frac{1}{2n} \int p_n' p_n e^{-x^2} \, dx \]

So \( p_n \) are orthogonal