① <u>Basic results on optimizers</u>

$f: U \to \mathbb{R}$, $U \subset \mathbb{R}^n$. Take $x^* \in U$; $B_r(x^*)$ — ball of radius r around x^*. We say that

- x^* is a <u>local optimizer</u> if for some $r>0$, $f(x^*) \le f(x)$, $x \in B_r(x^*) \subset U$; <u>strict</u> local optimizer, if $f(x^*) < f(x)$ for all $x \neq x^*$
- <u>global minimizer</u> if same true for U
- <u>critical point</u> if Gateaux diff. of f at x^* is 0
- <u>saddle point</u> if it's a critical pt & there are pts x', x''

$$f(x') < f(x^*) < f(x'') \text{ in any } B_r(x^*).$$

② Weierstraß' Theorem: if f is continuous on a compact set K. Then f possesses a global minimizer (or maximizer).

<u>Proof</u>: If a minimizer doesn't exist, we can find seq of points $\{x_k \in U\}$. $f(x_k) \to \inf_{x \in U} f$. But, by compactness of K, there is a subsequence (still, $f(x_k) \to \inf f$): $x_\ell \to x^* \in K$. Then $f(x^*) \le f(x)$, for all $x \in K$. $\square$

<u>Reminder</u> Compact sets: Main Definition: Open cover admits finite subcover
$\Rightarrow$ Any sequence has convergent subsequence
$\Rightarrow$ bounded & closed [in $\mathbb{R}^n$: $\Rightarrow$ compact]

A function on metric space D is called <u>coercive</u> if

$$f \to \infty \quad \text{as } |x| \to \infty \quad (|x| = \text{dist}(x, 0))$$

origin in $\mathbb{R}^n$; some basepoint in general

<u>Thm</u> If f is continuous coercive on closed subset of $\mathbb{R}^n$, then f has a global minimum.

<u>Proof</u>: $\{f \le c\}$ is either empty or bounded & closed. $\Rightarrow$ $\Rightarrow$ compact. $\Rightarrow$ has a minimum. $\square$

Application Any polynomial $p(z) = z^n + a_1 z^{n-1} + \dots + a_n$ has a root in $\mathbb{C}$.

Proof: $|f|$ is coercive, so it attains a global min at z_* (assume it $=0$). But if $p = a_0 + a_k z^k + a_{k+1} z^{k+1} + \dots$ (1st coefficient $\neq 0$)

then one can find v: $a_k v^k = \lambda a_0$, $\lambda < 0$, real.

For small v, $a_k z^k$ is much bigger than rest of terms $\Rightarrow$

$\Rightarrow |a_0 + a_k z^k + \dots| < |a_0| \dots$ $\square$

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② First order optimality conditions

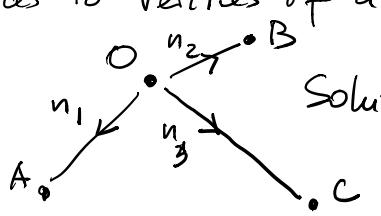
Mother of all optimality conditions:

Thm If $f: \mathcal{U} \to \mathbb{R}$ is Gâteaux differentiable on open $\mathcal{U}$, and x^* is a local minimizer, then $\nabla f(x^*) = 0$.

Proof: Taylor formula. $\square$

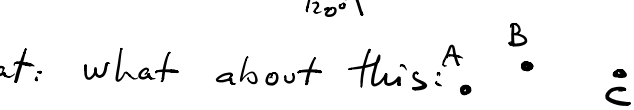
Some examples

① Steiner point: find point on the plane minimizing sum of distances to vertices of a triangle: $f(O) := |AO| + |BO| + |CO| \to \min_O$

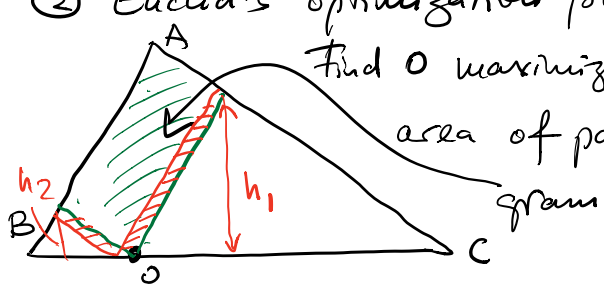


Solution: $\nabla f = \bar{n}_1 + \bar{n}_2 + \bar{n}_3 \Rightarrow$ "120° rule": at optimum,

Caveat: what about this:



② Euclid's optimization problem: Find O maximizing area of parallelogram



Solution: $\nabla f = h_2 - h_1$; so optimal when O is midpt between B & C

③

$$f(x_1 \dots x_n) = \frac{1}{2}\sum x_k^2 - \sum_{k\neq l} \ln|x_k - x_l|$$

Confining potential Coulomb repulsion

$f \to \min$. [Background: spectrum density for Gaussian Unitary Ensemble: random Hermitte matrix (reminder!) of size $n\times n$]

$$\nabla f = \left(\frac{\partial f}{\partial x_k}\right) = \left(x_k - \sum_{l\neq k} \frac{1}{x_k - x_l}\right)$$

Take $p = p_n = \prod_{k=1}^{n}(x - x_k)$ $p'(x_k) = \prod_{l\neq k}(x_k - x_l)$

$$\frac{p''}{p'}(x_k) = \frac{\sum_{l\neq k}\prod_{m\neq k,l}(x_k - x_m)}{\prod_{l\neq k}(x_k - x_l)} = 2\sum_{l\neq k}\frac{1}{x_k - x_l}$$

So, if (x_k) is minimizing, $\frac{p''}{p'}(x_k) = 2x_k$

$p''(x_k) - 2x_k p'(x_k) = 0$ for all x_k, roots of p.

$$\Longrightarrow \quad p_n''(x) - 2x p_n'(x) = -2n\, p_n(x)$$

NB: $\int_{-\infty}^{\infty} p_n p_m e^{-x^2}\,dx =$ $\left(p_n' e^{-x^2}\right)' = \left(p_n'' - 2x p_n'\right)e^{-x^2}$

$$= \int \left(\frac{2x p_n' - p_n''}{2n}\right) p_m e^{-x^2}\,dx =$$

$$= -\frac{1}{2n}\int \left(p_n' e^{-x^2}\right)' p_m\,dx = +\frac{1}{2n}\int p_n' p_m' e^{-x^2}\,dx$$

$= 0$ if $n \neq m$

So p_n are orthogonal