(1)
Of course, for example $f(x, y) = y\sin(x)$. All critical points $\{(\pi n, 0)\}, n \in \mathbb{Z}$, are saddles, as Hesse matrices are
$$\begin{pmatrix}
0 & \pm 1 \\
\pm 1 & 0
\end{pmatrix}.$$

(2)
Of course, for example $3x^2 + x^2\sin(1/x)$

(3)
The function $f(x, y) = (x^2 - 1)(y^2 - 1)(x^2 + 2xy + y^2 - 1)$ has 19 critical points. By setting $\frac{\partial f(x,y)}{\partial x} = 0$ and $\frac{\partial f(x,y)}{\partial y} = 0$, then solving for the critical points and checking the second-order condition, we find that there is one minimum, 6 maxima, and 12 saddles. The contour of the function is shown below.
Let \( A = (-2, 1), B = (2, 2), C = (3, -6), D = (-3, -3). \) Point 0, which is at the intersection of the diagonal \( AC\&BD \) is optimum. To see this, consider another point \( 0' \), by the triangular inequality,

\[
|A0'| + |0'C| \geq |AC| = |A0| + |C0|,
\]
\[
|B0'| + |0'D| \geq |BD| = |B0| + |D0|.
\]

\[(4)\]

We need to apply the use the first-order necessary condition on the function \( f(a, b, c) = \int_{-1}^{1} z^3 + az^2 + bz + c. \) Note that

\[
\frac{\partial f(a, b, c)}{\partial a} = 4\left(\frac{a}{3} + c\right) = 0,
\]
\[
\frac{\partial f(a, b, c)}{\partial b} = 4\left(\frac{b}{3} + \frac{a}{3}\right) = 0,
\]
\[
\frac{\partial f(a, b, c)}{\partial c} = 4\left(\frac{b}{3} + \frac{c}{3}\right) = 0,
\]

which implies that \( a = c = 0 \) and \( b = -\frac{3}{5}. \) Hence, \( \frac{2}{5}z \) is the best (mean quadratic) approximation to \( z^3 \) over \([-1, 1]\) among all quadratic polynomials.

\[(5)\]

Our function is \( (\sum_k v_k)^2 \geq 0. \) Hence, it attains 0 (its minimum) at any point where \( \sum_k v_k = 0 \) and \( \sum_k v_k^2 = 1 \) (e. g. \( v_1 = 1/\sqrt{2}, v_2 = -1/\sqrt{2}, v_{k>2} = 0 \)). By Cauchy-Schwartz,

\[
\left(\sum_k v_k^2\right) \left(\sum_k 1^2\right) \geq \left(\sum_k v_k 1\right)^2.
\]

On the unit sphere, LHS = 1, so, RHS is at most \( n \), which is attained at

\[ v_k = 1/\sqrt{n}, \ k = 1, \ldots, n. \]

Hence, the maximum of the quadratic form is \( n. \)