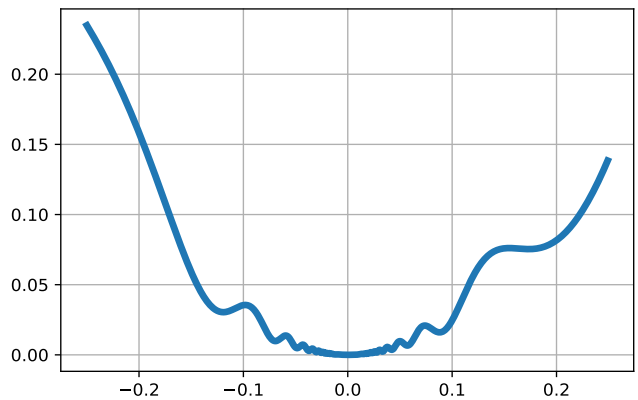
ECE 490: Introduction to Optimization (Homework I)	Spring 2019

(1)

Of course, for example $f(x, y) = y\sin(x)$. All critical points $\{(\pi n, 0)\}, n \in \mathbb{Z}$, are saddles, as Hesse matrices are $\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$.

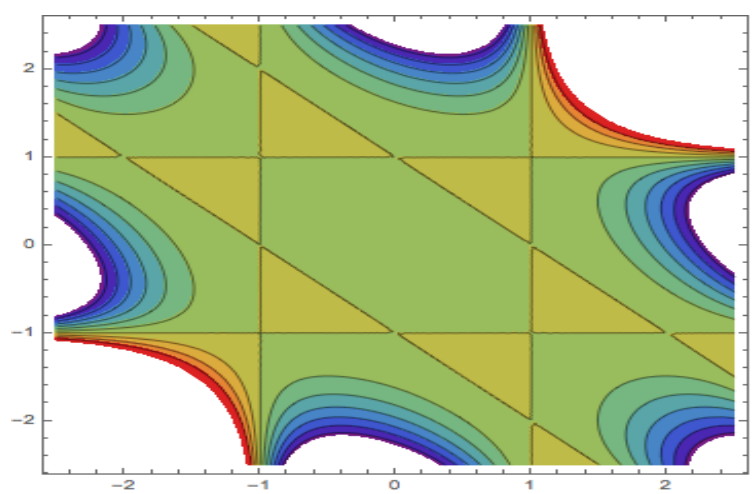
(2)

Of course, for example $3x^2 + x^2 \sin(1/x)$



(3)

The function $f(x, y) = (x^2-1)(y^2-1)(x^2+2xy+y^2-1)$ has 19 critical points. By setting $\frac{\partial f(x,y)}{\partial x} = 0$ and $\frac{\partial f(x,y)}{\partial y} = 0$, then solving for the critical points and checking the second-order condition, we find that there is one minimum, 6 maxima, and 12 saddles. The contour of the function is shown below.

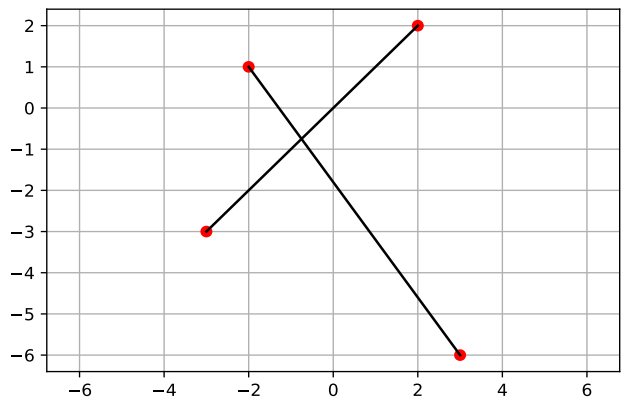


(4)

Let $A = (-2, 1), B = (2, 2), C = (3, -6), D = (-3, -3)$. Point 0, which is at the intersection of the diagonal $AC\&BD$ is optimum. To see this, consider another point $0'$, by the triangular inequality,

$$|A0'| + |0'C| \geq |AC| = |A0| + |C0|,$$

$$|B0'| + |0'D| \geq |BD| = |B0| + |D0|.$$



(5)

We need to apply the use the first-order necessary condition on the function $f(a, b, c) = \int_{-1}^{1} z^3 + az^2 + bz + c$. Note that

$$\begin{aligned}
\frac{\partial f(a,b,c)}{\partial a} &= 4(\tfrac{a}{3} + c) = 0, \\
\frac{\partial f(a,b,c)}{\partial b} &= 4(\tfrac{1}{5} + \tfrac{b}{3}) = 0, \\
\frac{\partial f(a,b,c)}{\partial c} &= 4(\tfrac{a}{5} + \tfrac{c}{3}) = 0,
\end{aligned}$$

which implies that $a = c = 0$ and $b = -\frac{3}{5}$. Hence, $\frac{3}{5}z$ is the best (mean quadratic) approximation to z^3 over $[-1, 1]$ among all quadratic polynomials.

(6)

Our function is $\left(\sum_k v_k\right)^2 \geq 0$. Hence, it attains 0 (its minimum) at any point where $\sum_k v_k = 0$ and $\sum_k v_k^2 = 1$ (e. g. $v_1 = 1/\sqrt{2}, v_2 = -1/\sqrt{2}, v_{k>2} = 0$). By Cauchy-Schwartz,

$$\left(\sum_k v_k^2\right)\left(\sum_k 1^2\right) \geq \left(\sum_k v_k 1\right)^2.$$

On the unit sphere, LHS $= 1$, so, RHS is at most n, which is attained at

$$v_k = 1/\sqrt{n}, \quad k = 1, \ldots, n.$$

Hence, the maximum of the quadratic form is n.