Problem 1 Consider the operator $A$ on $\mathbb{R}^N = \{(x_0, \ldots, x_{N-1})\}$ given by

$$(Ax)_n = x_{n-1} - 2x_n + x_{n+1},$$

where $n - 1, n + 1$ are understood mod$N$.

1. Find the eigenvalues of $A$.
2. Find $\exp(tA)v$, where

$$v_k = \exp(2\pi ikl/N), 0 \leq k < N$$

(here $l$ is an integer between 0 and $(n - 1)$).

Problem 2
Consider vector field in $\mathbb{R}^2$ given by

$$\dot{x}_1 = x_3 - 3x_1x_2^2 - x_1, \quad \dot{x}_2 = 3x_1^2x_2 - x_3^2 - x_2.$$

1. Find all stationary points of the vector field.
2. Linearize the vector field near the stationary points, and determine which of them are stable, asymptotically stable.

Problem 3
Find minimal realization of the system with transfer function

$$G(s) = \frac{1}{1 - s^3}$$

Problem 4
For the operator $A$ from Problem 1 (assume $N = 4$), set

$$\dot{x} = Ax + bu, b = (0, 0, 0, 1)^\ast.$$

1. Is the system controllable?
2. Find state feedback $k$ such that the closed loop operator has eigenvalues $(-1), (-2)$ (both eigenvalues of multiplicity 2).

Problem 5
Consider the LPR problem $\dot{x} = Ax + bu, x \in \mathbb{R}^2, J = \int_0^T |cx|^2 + r|u|^2dt$, where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, c = (1 \ 0), r = 1.$$

1. Solve algebraic Riccati equation.
2. Find the optimal state feedback for the infinite horizon, and determine the eigenvalues of the resulting closed loop operator.
Problem 6
Consider the travel time minimization problem

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_1 + u,
\end{align*}
\]

where \(|u| \leq 1|,

and the final position is (0,0).
Find for which starting positions \(x(0) = (x_1(0), x_2(0))\) the optimal time is finite.