Problem 1
Consider the matrix
\[
A = \begin{pmatrix}
0 & 1 & 1/2 & 0 \\
1 & 0 & 0 & -1/2 \\
1/2 & 0 & 0 & -1 \\
0 & -1/2 & -1 & 0
\end{pmatrix}.
\]

1. [5] Find characteristic polynomial and the eigenvalues of \(A\).

Solution Using your favorite software (I used Mathematica) we find
\[
p_A = 9/16 - 5/2s^2 + s^4, \sigma(A) = \{\pm 1/2, \pm 3/2\}.
\]

[5] Find bases for the kernel (null-space) and for the image (range) of operator \(4A^2 - E\), where \(E\) is the identity matrix.

Solution The kernel will be spanned by the eigenvectors corresponding to \(\pm 1/2\), i.e.
\[
v_{1/2} = (1, 1, -1, 1)^* \text{ and } v_{-1/2} = (1, -1, 1, 1)^*.
\]
(Of course, one can choose any basis within this 2-dimensional subspace.)
The image is spanned by any two columns of the matrix \(4A^2 - E\), e.g.
\[
v_+ = (0, 1, 1, 0)^* \text{ and } v_- = (1, 0, 0, -1)^*.
\]

[10] Find \(\exp(-A^2)\).

Solution The modal matrix diagonalizing \(A\) is
\[
M = \begin{pmatrix}
-1/2 & 1/2 & 1/2 & 1/2 \\
1/2, & -1/2 & 1/2 & 1/2 \\
1/2 & 1/2 & -1/2 & 1/2 \\
1/2 & 1/2 & 1/2, & -1/2
\end{pmatrix},
\]
and so
\[
\exp(-A^2) = \begin{pmatrix}
e^{-1/4}/2 + e^{-9/4}/2 & 0 & 0 & e^{-1/4}/2 - e^{-9/4}/2 \\
0 & e^{-1/4}/2 + e^{-9/4}/2 & 0 & 0 \\
0 & -e^{-1/4}/2 + e^{-9/4}/2 & e^{-1/4}/2 + e^{-9/4}/2 & 0 \\
e^{-1/4}/2 - e^{-9/4}/2 & 0 & 0 & e^{-1/4}/2 + e^{-9/4}/2
\end{pmatrix}
\]

Problem 2
[15] Find all points \((a, b)\) such that the restriction of the quadratic form \(q(x, y, z) := x^2 + y^2 - z^2\) to the plane \(V(a, b) \subset \mathbb{R}^3\) given by \(V(a, b) := \{z = ax + by\}\) is positive definite.

Solution Substituting \(z = ax + by\) into \(q\) yields the quadratic form
\[
q(x, y) = (1 - a^2)x^2 - 2abxy + (1 - b^2)y^2 = (x \quad y) \begin{pmatrix}
1 - a^2 & -ab \\
-ab & 1 - b^2
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix}.
\]
Sylvester criterion implies that it is positive definite iff 1) $1 - a^2 > 0$, and 2) 
\[(1 - a^2)(1 - b^2) - a^2b^2 = 1 - a^2 - b^2 > 0.\]
Hence, it is positive definite for $(a, b)$ within the unit circle.

**Problem 3**

1. [10] Determine the values of parameter $b$ for which all equilibria of the 1-dimensional dynamical system 

\[ \dot{x} = x - x^3 + b \]

are asymptotically stable.

**Solution** One finds critical points of the polynomial $x - x^3$ at $\pm \sqrt{3}/3$, and the values of the polynomial at these points are $\pm 2\sqrt{3}/9$. When $|b| > 2\sqrt{3}/9$, the vector field has unique stable equilibrium point; for all other $b$’s there is an unstable point.

[10] For each of the stable equilibria of the system $\dot{x} = x - x^3$ find their domains of stability;

**Solution** For $|b| > 2\sqrt{3}/9$, the whole line is the domain of stability of the unique equilibrium point. For $|b| < 2\sqrt{3}/9$, there are three roots, $r_- < r_0 < r_+$ of the polynomial $x - x^3 + b$. Two of them, $r_-$ and $r_+$ are asymptotically stable; their domains of stability are $(-\infty, r_0)$ and $(r_0, \infty)$, respectively. For $b = 0$, these roots are $r_- = -1, r_0 = 0, r_+ = 1$.

**Problem 4**

[10] Consider the system $\dot{x} = Ax + Bu, x \in \mathbb{R}^2, u \in \mathbb{R}$, where

\[ A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \]

Find all state feedbacks $K$ such that $A - BK$ is Hurwitz.

**Solution** For the gains $(k_1 \quad k_2)$, the closed loop operator is

\[ A_{cl} = \begin{pmatrix} 1 - k_1 & -1 - k_2 \\ 1 - k_1 & 1 - k_2 \end{pmatrix}, \]

so that the characteristic polynomial is

\[ p_{A_{cl}} = s^2 - (2 - k_1 - k_2)s + 2(1 - k_1). \]

The real parts of the roots are negative iff both of the coefficients are positive, so that the condition of $A_{cl}$ to be Hurwitz is

\[ k_1 + k_2 > 2; k_1 < 1. \]