Chapter 9

Tracking and regulation

9.1 The regulator problem

An important feedback synthesis problem is to design for a given control system a dynamic feedback controller such that the output of the resulting closed-loop system *tracks* (i.e., converges to), some a priori given reference signal. This problem is known as the *servo problem*.

In the case that the reference signal is equal to a Bohl function from a certain time on (which covers the important special cases that the reference signal is, for example, a step function, a ramp function or a sinusoid), one way to approach the servo problem is to let the reference signal be generated by some dynamical model, more specifically, to set up some linear, time invariant, autonomous system that, for some appropriate initial state, has the reference signal as its output. Note that the frequencies of this reference signal are fixed by the dynamics of this autonomous system (to be called *the exosystem*) while phase and amplitude of the different frequencies is determined by the initial condition of this exosystem. One then incorporates the equations of this exosystem into the equations of the control system, and defines a new output as the difference between the outputs of the exosystem and the control system. The servo problem can then be reformulated as: design a dynamic feedback controller such that the output of the aggregated system converges to zero regardless of its initial state. In particular, by taking the appropriate initial state for the exosystem, the deviation of the output from the reference signal (called the *tracking error*) will then converge to zero as time tends to infinity.

A second important synthesis problem is the problem of *output regulation*. For a certain control system that is subjected to external disturbances, the problem is here to design a dynamic feedback controller such that the output of the closed-loop system converges to zero as time tends to infinity, regardless of the disturbance and the initial state. One way to approach this problem is to consider the disturbances to be completely unknown, but to be elements of some function class **D** (in fact, this setup is worked out in exercise 6.4). In this chapter, we will take an alternative point of view and assume the disturbances, albeit unknown, to be generated as outputs of some linear time-invariant autonomous system, again to be called the *exosystem*. This basically amounts to the fact that the function class **D** dictates that there are only a fixed set of frequencies in the disturbance signal. Each initial state of the exosystem then corresponds to one disturbance function and this initial state fixes the phase and amplitude of each frequency. One incorporates the equations of the exosystem into the equations of the control system, and requires the output of the new, aggregated, system to converge to zero (to be *regulated*), regardless of the initial state.

Of course, an even more general problem formulation is obtained by combining these two synthesis problems into a single one by requiring the design of a dynamic feedback controller such that the output of the closed loop system tracks a given reference signal, regardless of the disturbance and the initial state. It should be clear that this combined problem can be approached by combining the two exosystems into a single one and to require regulation of the tracking error.

As an illustration, consider a scalar control system whose output is required to track a sinusoid, in the presence of constant disturbances. Let the control system be given by

$$\dot{x}_1(t) = a_{11}x_1(t) + b_1u(t) + a_{14}d(t), \quad z_1(t) = x_1(t).$$

Suppose the reference signal is $r(t) = \sin \omega t$. This reference signal can be generated by the system

$$\begin{aligned} \dot{x}_2(t) &= x_3(t), \\ \dot{x}_3(t) &= -\omega^2 x_2(t), \\ r(t) &= x_2(t), \end{aligned}$$

by taking the initial conditions $x_2(0) = 0$ and $x_3(0) = \omega$. The tracking error is equal to $z_1(t) - r(t)$. Suppose that the disturbances d are known to be constant, but with unknown magnitude. This can be modelled by letting the disturbances be generated by

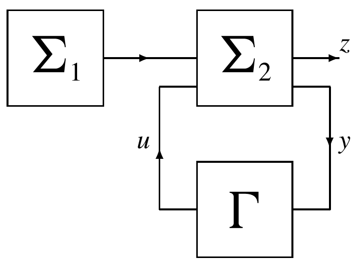
$$\begin{aligned} \dot{x}_4(t) &= 0, \\ d(t) &= x_4(t). \end{aligned}$$

Both reference signal and disturbance signals can be thought of as being generated by a single exosystem, obtained by combining the respective equations. The aggregated system is then given by

$$\begin{aligned} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{pmatrix} &= \begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & 0 & 1 & 0 \\ 0 & -\omega^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} + \begin{pmatrix} b_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} u(t), \\ z(t) &= \begin{pmatrix} 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix}. \end{aligned}$$

In addition to the requirements of tracking and regulation, a realistic design requires the property of internal stability. Obviously, one can not expect to be able to internally stabilize the aggregated system, since typically part of this system (the exosystem) can not be influenced by controls and will generally be unstable. Thus, in the present context the requirement of internal stabilization should be interpreted as internal stabilization of the interconnection of the original control system and the designed controller.

We will now make things more precise. Suppose that we are given a control system which is subject to a disturbance of a specified type, and whose output should track a given reference signal. This situation is modelled as the interconnection of two systems, Σ_1 and Σ_2, where Σ_2 denotes the control system and Σ_1 an autonomous system that generates the disturbances and the reference signal, called the *exosystem*.



It is assumed that the control system also has a control input u and two outputs y and z, as in the previous chapters. Also, we assume that a stability domain $\mathbb{C}_g$ has been prescribed. The regulator problem then consists of finding a controller Γ such that for the resulting closed loop system the following properties hold:

- *the regulation property*: $z(t)$ is $\mathbb{C}_g$-stable for any initial state of the total closed loop system.
- *internal stability*: for zero initial state of the exosystem and any initial state of the control system and the controller, the combined state of the control system and the controller is $\mathbb{C}_g$-stable.

Again, we note that we can not hope to achieve internal stability of the total closed loop system, since the exosystem is completely uncontrollable and typically unstable. If it happens to be internally stable, the problem reduces to the classical stabilization problem, treated in section 3.12. In fact, one usually assumes that the exosystem is *antistable* (see definition 2.12).

Let us now specify the system considered. We shall assume that the exosystem Σ_1 is given by the equation

$$\dot{x}_1(t) = A_1 x_1(t), \tag{9.1}$$

while the plant Σ_2 is assumed to be given by the equations

$$\begin{aligned} \dot{x}_2(t) &= A_3 x_1(t) + A_2 x_2(t) + B_2 u(t), \\ y(t) &= C_1 x_1(t) + C_2 x_2(t), \\ z(t) &= D_1 x_1(t) + D_2 x_2(t) + Eu(t). \end{aligned} \tag{9.2}$$

The state space $\mathcal{X}_2$ of Σ_2 is assumed to be n_2-dimensional. The output space $\mathcal{Z}$ is r-dimensional. The disturbance enters the plant via the term $A_3 x_1$ in the state equation and via the terms $C_1 x_1$ and $D_1 x_1$ in the output equations. We allow for a direct feedthrough term Eu from the control input to the to-be-controlled variable. Such a term is omitted in the equation for y, because it would have been inconsequential for the present problem.

It is convenient to combine Σ_1 and Σ_2 according to equations (9.1) and (9.2) to one system Σ with state variable $x = (x_1^{\mathrm{T}}, x_2^{\mathrm{T}})^{\mathrm{T}}$ and coefficient matrices

$$\begin{aligned} &A := \begin{pmatrix} A_1 & 0 \\ A_3 & A_2 \end{pmatrix}, \ B := \begin{pmatrix} 0 \\ B_2 \end{pmatrix}, \ C := \begin{pmatrix} C_1 & C_2 \end{pmatrix}, \\ &D := \begin{pmatrix} D_1 & D_2 \end{pmatrix}, \end{aligned} \tag{9.3}$$

so that we have the following equations for Σ :

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \\ z(t) &= Dx(t) + Eu(t). \end{aligned} \tag{9.4}$$

Before investigating the existence of a controller with the desired properties, we describe what systems already satisfy the regulation property and the internal stability condition. We define these properties for systems without control inputs, equivalently, for $u = 0$. We say that Σ is *endostable* if Σ_2 is internally stable. This means that for $x_1(0) = 0$ and arbitrary $x_2(0)$, the state $x_2(t)$ is $\mathbb{C}_g$-stable. We say that Σ is (*output*) *regulated* if $z(t)$ is $\mathbb{C}_g$-stable for every initial state of Σ. These properties can be expressed in terms of the coefficient matrices. Obviously, Σ is endostable if and only if $\sigma(A_2) \subset \mathbb{C}_g$. For output regulation, we have the following result.

Lemma 9.1 *Assume that $\sigma(A_2) \subset \mathbb{C}_g$. Then the system Σ with input $u = 0$ is output regulated if the equations*

$$\begin{aligned} TA_1 - A_2 T &= A_3, \\ D_2 T + D_1 &= 0 \end{aligned} \tag{9.5}$$

in T are solvable. If A_1 is antistable, this condition is also necessary.

If A_1 is not antistable then we can delete the stable part since it does not effect regulation which is only an asymptotic condition. In this way we can basically reduce the general problem to the case when A_1 is antistable.

Proof : The main idea is that for large t, the state x_2 of the plant is close to Tx_1, for a suitable linear map T. So we introduce the variable $v := x_2 - Tx_1$, where we specify

T later on. In a straightforward calculation, one derives from the equations (9.1) and (9.2) that

$$\begin{aligned} \dot{v}(t) &= A_2 v(t) + (A_2 T - T A_1 + A_3) x_1(t)\,, \\ z(t) &= D_2 v(t) + (D_1 + D_2 T) x_1(t)\,. \end{aligned} \tag{9.6}$$

Now assume that T is a solution of the equations (9.5). Then the equations (9.6) reduce to

$$\begin{aligned} \dot{v}(t) &= A_2 v(t), \\ z(t) &= D_2 v(t). \end{aligned}$$

Since A_2 is $\mathbb{C}_g$-stable, theorem 3.23 implies that $z(t)$ is $\mathbb{C}_g$-stable.

Conversely, assume that A_1 is antistable and that the system Σ is regulated and endostable. Then it follows from Sylvester's Theorem (see section 9.3) that there exists a (unique) matrix T satisfying the first equation of (9.5). Substituting this into (9.6), we find that $v(t)$ is $\mathbb{C}_g$-stable. Since $z(t)$ is also $\mathbb{C}_g$-stable, this implies that $(D_2 T + D_1) x_1(t)$ is $\mathbb{C}_g$-stable. However, because x_1 is antistable for any initial condition, we must have that $D_2 T + D_1 = 0$. ■

Now we want to solve the regulator problem by constructing a controller such that the closed loop system satisfies the conditions of lemma 9.1. As usual, the controller Γ will be of the form

$$\begin{aligned} \dot{w}(t) &= K w(t) + L y(t), \\ u(t) &= M w(t) + N y(t). \end{aligned} \tag{9.7}$$

The closed loop system will be equal to the cascade connection Σ_{cl} of Σ_1 and $\Sigma_{2,cl}$, where $\Sigma_{2,cl}$ is the feedback interconnection of Σ_2 and Γ, given by

$$\begin{aligned} \dot{x}_{2e}(t) &= A_{2e} x_{2e}(t) + A_{3e} x_1(t), \\ z(t) &= D_{1e} x_1(t) + D_{2e} x_{2e}(t), \end{aligned}$$

where

$$\begin{aligned} A_{2e} &:= \begin{pmatrix} A_2 + B_2 N C_2 & B_2 M \\ L C_2 & K \end{pmatrix}, \qquad A_{3e} := \begin{pmatrix} A_3 + B_2 N C_1 \\ L C_1 \end{pmatrix}, \\ D_{2e} &:= \begin{pmatrix} D_2 + E N C_2 & E M \end{pmatrix}, \qquad D_{1e} := D_1 + E N C_1. \end{aligned} \tag{9.8}$$

We call Γ a *regulator* if Σ_{cl} is endostable and output regulated. The problem of finding a regulator will be called the *regulator problem*. It follows from lemma 9.1 that the regulator problem can be solved by finding $\Gamma = (K, L, M, N)$ such that A_{2e} is stable and the equations

$$T_e A_1 - A_{2e} T_e = A_{3e}\ , \ \ D_{2e} T_e + D_{1e} = 0 \tag{9.9}$$

have a solution T_e. The existence of a solution T_e is necessary for the existence of a regulator if A_1 is antistable. In order to be able to solve this problem we shall make two assumptions:

- Σ_2 is stabilizable with u as input, i.e., (A_2, B_2) is stabilizable.
- Σ is detectable with y as output, i.e., (C, A) is detectable

Obviously, the stabilizability of Σ_2 is necessary for the existence of a regulator. Also the detectability of Σ_2 is necessary. However, here we impose a more restrictive condition on the system, viz. the detectability of the total system Σ. Note that there exists a standard reduction technique to solve the problem when Σ_2 is detectable but Σ is not detectable. In an appropriate manner, this technique deletes the undetectable modes and only requires us to design a suitable regulator for the remaining system which satisfies the stronger condition of detectability we impose in this chapter. For details we refer to [44,159]. Then we have the following result:

Theorem 9.2 *Assume that (A_2, B_2) is stabilizable and (C, A) is detectable. Then there exists a regulator if the equations*

$$\begin{aligned} TA_1 - A_2T - B_2V &= A_3, \\ D_1 + D_2T + EV &= 0 \end{aligned} \tag{9.10}$$

have a solution (T, V). If A_1 is antistable, the solvability of (9.10) is necessary for the existence of a regulator. Specifically, if $G : \mathcal{Y} \to \mathcal{X}$ is such that $\sigma(A + GC) \subset \mathbb{C}_g$, $F_2 : \mathcal{X} \to \mathcal{U}$ is such that $\sigma(A_2 + B_2F_2) \subset \mathbb{C}_g$ and if $F_1 := -F_2T + V$, $F := (F_1 \;\; F_2)$, where (T, V) is a solution of (9.10), then a regulator is given by

$$\begin{aligned} \dot{w}(t) &= (A + GC + BF)w(t) - Gy(t), \\ u(t) &= Fw(t). \end{aligned} \tag{9.11}$$

Proof : Assume that A_1 is antistable and that a regulator exists. This regulator satisfies (9.9) for some T_e. We decompose T_e into $T_e = (T^{\mathrm{T}}, U^{\mathrm{T}})^{\mathrm{T}}$ and substitute (9.8) into (9.9). The first block row of the first of the resulting equations reads:

$$TA_1 - (A_2 + B_2NC_2)T - B_2MU = A_3 + B_2NC_1$$

and the second equation reads:

$$D_1 + (D_2 + ENC_2)T + EMU + ENC_1 = 0.$$

These relations show that $(T, NC_2T + MU + NC_1)$ is a solution of (9.10).

Conversely, assume that (T, V) satisfies (9.10). Define a controller Γ by

$$(K, L, M, N) := (A + GC + BF, -G, F, 0)$$

i.e., by (9.11) where F and G satisfy the conditions of theorem 9.2. We claim that Γ is a regulator. To show this, we have to prove that the resulting extended system is endostable, and that we have the regulation property. In order to prove endostability, we introduce $r := w - x$ and notice that x and r satisfy

$$\begin{aligned} \dot{x}(t) &= (A + BF)x(t) + BFr(t) \;, \\ \dot{r}(t) &= (A + GC)r(t) \;\;. \end{aligned}$$

Obviously, $r(t)$ is $\mathbb{C}_g$-stable. If $x_1(0) = 0$ (and hence $x_1(t) = 0$ for all t), the first equation reduces to

$$\dot{x}_2(t) = (A_2 + B_2F_2)x_2(t) + B_2Fr(t).$$

Then we also have $x_2(t)$ is $\mathbb{C}_g$-stable (compare theorem 2.7). Next we verify that Σ_e is output regulated. To this extent, we define $U := (I, T^{\mathrm{T}})^{\mathrm{T}}$ and we claim that $T_e := (T^{\mathrm{T}}, U^{\mathrm{T}})^{\mathrm{T}}$ satisfies (9.9). To show this, we substitute this and (9.8) into (9.9). Then for the first equation of (9.9) we have to prove that

$$\begin{pmatrix} T \\ U \end{pmatrix} A_1 - \begin{pmatrix} A_2 & B_2F \\ -GC_2 & A + GC + BF \end{pmatrix} \begin{pmatrix} T \\ U \end{pmatrix} = \begin{pmatrix} A_3 \\ -GC_1 \end{pmatrix}.$$

We notice that $FU = V$. Hence the first block equation is exactly the first equation of (9.10), viz. $TA_1 - A_2T - BV = A_3$. The second equation takes some more effort. It reads:

$$G(C_1 + C_2T - CU) + UA_1 - AU - BV = 0.$$

The expression between parentheses equals zero because of the definition of U. The remaining terms are decomposed according to (9.3):

$$\begin{pmatrix} I \\ T \end{pmatrix} A_1 - \begin{pmatrix} A_1 & 0 \\ A_3 & A_2 \end{pmatrix} \begin{pmatrix} I \\ T \end{pmatrix} - \begin{pmatrix} 0 \\ B_2 \end{pmatrix} V = \begin{pmatrix} A_1 - A_1 \\ TA_1 - A_3 - A_2T - B_2V \end{pmatrix} = 0,$$

where we have used the first equation of (9.10). This shows that the first equation of (9.9) is satisfied.

Next we consider the second equation of (9.9). It reads $D_1 + D_2T + EV = 0$, which is the same as the second equation of (9.10) and hence it is immediately clear that this is also satisfied. ■

9.2 Well-posedness of the regulator problem

A mathematical problem is called *well posed* if it is solvable and it remains solvable after a small perturbation of the data of the problem. The equation $x^2 + y^2 - ay + b = 0$, for example, is solvable (in $\mathbb{R}^2$) for $a = 2$ and $b = 1$, but it is not well posed for these values of a and b because the solvability is lost when b is replaced by $1 + \varepsilon$ for arbitrary $\varepsilon > 0$. The investigation of the well-posedness is easy for linear equations. As a matter of fact, we have

Lemma 9.3 *Let $\mathcal{X}$ and $\mathcal{Y}$ be finite-dimensional linear spaces and let $A : \mathcal{X} \to \mathcal{Y}$ be a linear map and $b \in \mathcal{Y}$. Then the equation $Ax = b$ in the variable x is well posed if and only if A is surjective.*

Proof : If A is surjective, any matrix representation of A will have a nonzero subdeterminant of dimension equal to the number of rows. Since this determinant is a continuous function of the entries of the matrix (in fact, a polynomial) it follows that it will remain nonzero when the entries are perturbed a little bit. Hence, A will remain surjective, after a small perturbation. Therefore, the equation $Ax = b$ remains solvable for small perturbations of A and b. Obviously, b can be perturbed in an arbitrary way and not just locally.

Conversely, if A is not surjective, $\operatorname{im} A$ is a proper subspace of $\mathcal{Y}$. The equation $Ax = b$ is solvable if and only if $b \in \operatorname{im} A$. However, b cannot be an interior point of $\operatorname{im} A$, since $\operatorname{im} A$ contains no interior points. Consequently, an arbitrary small perturbation of b may take it out of $\operatorname{im} A$ and hence destroy the solvability of the equation. ■

Remark 9.4 In concrete situations, it is of importance to specify more precisely what the 'data' of the problem is. Sometimes not all of the entries in a matrix are considered data and subject to perturbations. For example, if A is a companion matrix, only the last row is considered data. The remaining entries consist of 'hard' zeros and ones. It follows from the above proof that the necessity of the surjectivity of A still holds if only the vector b is considered data.

Remark 9.5 Usually, and in particular in the case of lemma 9.3, the well-posedness problem is easier to solve than the original equation. It is easier to verify the surjectivity of a map A than to examine the solvability of the equation $Ax = b$.

We want to apply the above result to the regulator problem. For the solvability of the regulator problem, a number of conditions are imposed. In the first place, it is assumed that Σ_2 is stabilizable and Σ detectable. It is not difficult to see that these properties are invariant under small perturbations. For instance, if (A, B) is stabilizable and F is a stabilizing feedback, then $A + BF$ is a stability matrix. Since the eigenvalues depend continuously on the matrix, $\bar{A} + \bar{B}F$ is also a stability matrix if $\bar{A}$ and $\bar{B}$ are close to A and B. The important part to check is the well-posedness of the matrix linear equation (9.10). For this, we apply the previous theorem. We find that the equation:

$$\begin{aligned} TA_1 - A_2T - B_2V &= A_3, \\ D_2T + EV &= -D_1 \end{aligned}$$

in the variables T and V is well posed if and only if the map

$$(T, V) \mapsto (TA_1 - A_2T - B_2V, D_2T + EV)$$

is surjective. In order to check this condition, one can give a matrix representation of this map using tensor products. We will however follow a different procedure, based on general considerations on the solvability of matrix equations. The advantage of the condition thus obtained will be that it can be interpreted in systemic terms, specifically, in terms of the zeros of a system.

9.3 Linear matrix equations

The subject of this section is the solvability of linear matrix equations of the form

$$\sum_{i=1}^{k} A_i X B_i = C, \tag{9.12}$$

where A_i, B_i and C are given matrices and X is unknown. We distinguish between universal and individual solvability of (9.12). We say that (9.12) is *universally* solvable if the equation has a solution for every C. Universal solvability thus is a condition on the matrices A_i and B_i. If we want to stress that the equation is solvable for the particular C given, we say that (9.12) is *individually* solvable. Conditions for solvability can be given by viewing the left-hand side of (9.12) as a linear map L acting on the matrix X. Then (9.12) is individually solvable if and only if $C \in \operatorname{im} L$, and (9.12) is universally solvable if and only if L is surjective. One can give explicit conditions for these properties using tensor (or Kronecker) products, but this will give rise to huge matrices and little insight. Rather, we would like to have results in the spirit of *Sylvester's theorem*: *the equation $AX - XB = C$, where A and B are square matrices, is universally solvable if and only if $\sigma(A) \cap \sigma(B) = \varnothing$.* It does not seem possible to obtain a similar result for the general equation (9.12). However, if we restrict ourselves to the case where the matrices B_i are of the form $B_i = q_i(B)$ for given polynomials q_i and a fixed matrix B, we can derive the following:

Theorem 9.6 *Let $A_i \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{p\times p}$, and let $q_i(s)$ be polynomials for $i = 1, \ldots, k$. Let*

$$A(s) := \sum_{i=1}^{k} A_i q_i(s). \tag{9.13}$$

Then the equation

$$\sum_{i=1}^{k} A_i X q_i(B) = C, \tag{9.14}$$

is universally solvable if and only if $\operatorname{rank} A(\lambda) = n$ *for all* $\lambda \in \sigma(B)$.

It is straightforward that Sylvester's theorem is a special case of this result.

Proof : ('only if':) Suppose that $\operatorname{rank} A(\lambda) < n$ for some $\lambda \in \sigma(B)$. Choose nonzero vectors v and w such that $Bv = \lambda v$ and $w^{\mathrm{T}} A(\lambda) = 0$. Then we have for any matrix X:

$$w^{\mathrm{T}} \sum A_i X q_i(B) v = w^{\mathrm{T}} \sum A_i X q_i(\lambda) v = w^{\mathrm{T}} A(\lambda) X v = 0.$$

Hence, if $w^{\mathrm{T}} C v \neq 0$ (e.g. if $C = wv^{\mathrm{T}}$), (9.14) does not have a solution.

('if':) The polynomial matrix $A(s)$ is obviously right invertible as a rational matrix. Hence there exists a polynomial matrix $D(s)$ and a scalar polynomial $a(s)$ such that $A(s)D(s) = a(s)I$, where $a(s)$ is the product of the invariant factors of $A(s)$. The assumption of the theorem implies that $a(\lambda) \neq 0$ for $\lambda \in \sigma(B)$, hence, by the spectral mapping theorem (see (2.11)), that $a(B)$ is nonsingular. Defining $C_1 := C(a(B))^{-1}$ and $E(s) := D(s)C_1$ we find

$$\sum_{i=1}^{k} A_i E(s) q_i(s) = A(s)E(s) = C_1 a(s).$$

Next we apply right substitution of $s = B$ into this equation and use theorem 7.6. This yields $\sum_{i=1}^{k} A_i E_r(B) q_i(B) = C_1 a(B) = C$, which shows that $X := E_r(B)$ (the index r indicates right substitution) is a solution of (9.14). ■

Next we investigate the individual solvability of equation (9.14). For the equations $AX - XB = C$ and $AX - YB = C$, such conditions were given by Roth in 1952, viz.

Theorem 9.7

(i) *Let A, B and C be matrices such that the equation $AX - XB = C$ is defined. Then this equation has a solution if and only if the matrices*

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

are similar.

(ii) *If A, B and C are polynomial matrices of dimensions such that the equations $AX - YB = C$ makes sense, then this equation has a (polynomial matrix) solution if and only if*

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

are unimodularly equivalent.

An elegant proof of these theorem can be found in [80, Theorem 4.4.22]. We want to generalize this result to the equation (9.14). A generalization in terms of similarity seems difficult. However, it is known that two matrices A and $\bar{A}$ are similar if and only if the polynomial matrices $sI - A$ and $sI - \bar{A}$ are unimodularly equivalent. Hence, according to Roth, the equation $AX - XB = C$ is solvable if and only if the polynomial matrices

$$\begin{pmatrix} sI - A & 0 \\ 0 & sI - B \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} sI - A & C \\ 0 & sI - B \end{pmatrix}$$

are unimodularly equivalent. This formulation has a direct generalization:

Theorem 9.8 *Let A_i, B, $q_i(s)$ and $A(s)$ be as in theorem 9.6. Then equation (9.14) is (individually) solvable if and only if the matrices*

$$\begin{pmatrix} A(s) & 0 \\ 0 & sI - B \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} A(s) & C \\ 0 & sI - B \end{pmatrix} \tag{9.15}$$

are unimodularly equivalent.

Proof : First we note that (9.14) has a solution if and only if the equation

$$A(s)P(s) + Q(s)(sI - B) = C \tag{9.16}$$

in the polynomial matrices $P(s)$, $Q(s)$ has a solution. In fact, if (9.16) has a solution, we apply right substitution of $s = B$ into this equation, yielding (9.14) with $X = P_r(B)$ (the index r indicates right substitution). Conversely, let X be a solution of (9.14). Then we write:

$$C - A(s)X = \sum_i A_i X(q_i(B) - q_i(s)I) = \sum_i A_i V_i(s)(sI - B)$$

for certain polynomial matrices $V_i(s)$. Hence $(P(s), Q(s)) := (X, \sum A_i V_i(s))$ is a solution of (9.16). Next we notice that equation (9.16) is an equation of the type given in theorem 9.7 (ii). Hence (9.16) is solvable if and only if the two matrices in (9.15) are unimodularly equivalent. ■

9.4 The regulator problem revisited

In theorem 9.2, we saw that subject to the assumptions that Σ_2 is stabilizable and Σ is detectable, a sufficient condition for the existence of a regulator is the solvability of the matrix equation (9.10). In the present section, we intend to apply the results of section 9.3. To this extent, we rewrite (9.10) to

$$\begin{pmatrix} -A_2 & -B_2 \\ D_2 & E \end{pmatrix} \begin{pmatrix} T \\ V \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ V \end{pmatrix} A_1 = \begin{pmatrix} A_3 \\ -D_1 \end{pmatrix}. \tag{9.17}$$

This is an equation of the form (9.14). The solvability of this equation is not affected if the right-hand side is multiplied by -1. After this, the matrices defined in theorem 9.6 reduce to

$$A(s) = \begin{pmatrix} sI - A_2 & -B_2 \\ D_2 & E \end{pmatrix}, \quad B = A_1, \quad C = \begin{pmatrix} -A_3 \\ D_1 \end{pmatrix}. \tag{9.18}$$

Hence according to theorem 9.8, equation (9.17) has a solution if and only if the matrices

$$P(s) := \begin{pmatrix} sI - A_1 & 0 & 0 \\ -A_3 & sI - A_2 & -B_2 \\ D_1 & D_2 & E \end{pmatrix}$$

and

$$P_{\text{disc}}(s) := \begin{pmatrix} sI - A_1 & 0 & 0 \\ 0 & sI - A_2 & -B_2 \\ 0 & D_2 & E \end{pmatrix}$$

are unimodularly equivalent. Here we have applied an obvious row and column operation. Note that $P(s)$ is the system matrix as defined in section 7.2. Also $P_{\text{disc}}(s)$ can be interpreted as a system matrix, viz. the system matrix of the disconnected system Σ_{disc} obtained from Σ by disconnecting Σ_1 and Σ_2, i.e., by setting $A_3 = 0$, $D_1 = 0$. Recall that two polynomial matrices are unimodularly equivalent if and only if they have the same invariant factors (corollary 7.3). The invariant factors of the system matrix are defined to be the transmission polynomials of the system. Hence we have found the following:

Theorem 9.9 *Assume that (A_2, B_2) is stabilizable and that (C, A) is detectable. Then there exists a regulator for Σ if Σ and Σ_{disc} have the same transmission polynomials. If A_1 is antistable then this condition is also necessary.*

Now we investigate the well-posedness of the regulator problem. As was shown in section 9.2, this is guaranteed if the equation (9.10) or, equivalently, (9.17) is well posed (assuming that Σ_2 is stabilizable and Σ is detectable. Recall that these conditions are well posed). Hence, applying theorem 9.6, we find the following result:

Theorem 9.10 *Assume that (A_2, B_2) is stabilizable and (C, A) is detectable. Then the regulator problem is well posed if*

$$\operatorname{rank} \begin{pmatrix} \lambda I - A_2 & -B_2 \\ D_2 & E \end{pmatrix} = n_2 + r$$

(i.e., of full row rank) for every $\lambda \in \sigma(A_1)$. If A_1 is antistable this condition is also necessary.

In system-theoretic terms this condition requires that Σ_2 is *right-invertible and its zeros do not coincide with poles of* Σ_1 (for the notion of right-invertibility we refer to chapter 8). The necessary and sufficient conditions of this section are easily extended to the case where A_1 is not antistable. We omit the details. (See also exercise 9.2)

9.5 Exercises

9.1 Consider the system given as the interconnection of the exosystem

$$\begin{aligned} \dot{x}_1(t) &= -\omega x_2(t), \\ \dot{x}_2(t) &= \omega x_1(t), \end{aligned}$$

and the control system

$$\begin{aligned}
\dot{x}_3(t) &= -x_3(t) + x_5(t) + a x_1(t),\\
\dot{x}_4(t) &= x_5(t),\\
\dot{x}_5(t) &= x_3(t) + 3x_4(t) + 2x_5(t) + u(t),\\
y(t) &= \begin{pmatrix} x_2(t)\\ x_3(t)\\ x_4(t)\\ x_5(t)\end{pmatrix},\\
z(t) &= x_3(t) - x_1(t),
\end{aligned}$$

Assume that $\mathbb{C}_g = \mathbb{C}^-$.

a. For which values of a and ω is the regulator problem well posed?

b. Construct a regulator.

9.2 Consider the regulator problem without the assumption that A_1 is antistable. Show that if (A_2, B_2) is stabilizable and (C, A) is detectable, then the regulator problem is well posed if and only if

$$\operatorname{rank}\begin{pmatrix} \lambda I - A_2 & -B_2\\ D_2 & E\end{pmatrix} = n_2.$$

for every $\lambda \in \sigma(A_1) \cap \mathbb{C}_b$.

9.3 Let $A \in \mathbb{R}^{n\times n}$ and $F(s)$ be an $n \times m$ polynomial matrix. Show that

$$\operatorname{rank}(\lambda I - A \;\; F(\lambda)) = n$$

for all $\lambda \in \mathbb{C}$ if and only if $(A, F_\ell(A))$ is controllable. Here, $F_\ell(A)$ denotes the result of substituting $s = A$ into $F(s)$ from the left.

9.4 Show that (A, B) is controllable if and only if for every $n \times n$ matrix C there exist matrices X and U (of suitable dimensions) such that $XA - AX + BU = C$.

9.5 In this problem we consider the regulator problem. Let the exosystem be given by the equation

$$\Sigma_1 : \dot{x}_1 = \alpha_1 x_1,$$

with $x_1(t) \in \mathbb{R}$, $\alpha_1 \geqslant 0$. In addition, let the plant be given by

$$\Sigma_2 : \begin{aligned} \dot{x}_2 &= \quad a_3 x_1 + A_2 x_2 + B_2 u,\\ y &= z = d_1 x_1 + D_2 x_2,\end{aligned}$$

with $x_2(t) \in \mathbb{R}^{n_2}$ (we write a_3, d_1 instead of A_3, D_1 to stress that these matrices consist of one column). Assume that (A_2, B_2) is $\mathbb{C}^-$-stabilizable and that (D, A) is $\mathbb{C}^-$-detectable where

$$A := \begin{pmatrix} \alpha_1 & 0\\ a_3 & A_2\end{pmatrix};\; D := (d_1 \;\; D_2).$$

a. Show that if

$$\Gamma : \begin{array}{l} \dot{w} = Kw + Ly, \\ u \ = Mw + Ny, \end{array}$$

is a regulator, then there exists vectors t_0 and u_0 such that

$$\begin{array}{l} (\alpha_1 I - A_2)t_0 - B_2 M u_0 = a_3, \\ K u_0 = \alpha_1 u_0, \\ D_2 t_0 + d_1 = 0. \end{array}$$

b. Show that if Γ is a regulator then α_1 is an eigenvalue of K.

c. Now let the exosystem be given by

$$\Sigma_1 : \dot{x}_1 = A_1 x_1,$$

with $x_1(t) \in \mathbb{R}^{n_1}$, A_1 anti-stable. In addition, assume that the plant is given by

$$\Sigma_2 : \begin{array}{lll} \dot{x}_2 = & & A_2 x_2 + A_3 x_1 + B_2 u, \\ y & = z = & D_1 x_1 + D_2 x_2. \end{array}$$

Again assume that (A_2, B_2) is $\mathbb{C}^-$-stabilizable and that (D, A) is $\mathbb{C}^-$-detectable, with D and A defined as usual. Use the ideas from a) and b) to show that if Γ is a regulator, then we have $\sigma(A_1) \subset \sigma(K)$.

The phenomenon illustrated in this problem is an example of the famous *internal model principle*: the set of eigenvalues $\sigma(A_1)$ of the exosystem is contained in the set of eigenvalues $\sigma(K)$ of the regulator: in a sense, the regulator contains an internal model of the exosystem.

9.6 Notes and references

The regulator problem has been studied by many people. See for instance Davison in [35], Davison and Goldenberg in [36], Francis in [44], Francis and Wonham in [48] and Desoer and Wang in [38]. The theory has also been extended to for instance nonlinear systems by Isidori and Byrnes in [87]. Many results have recently been collected by Saberi, Stoorvogel and Sannuti in the book [159]. The regulator equations (9.9) were originally introduced by Francis in [45].

The techniques presented in section 9.3 can be found in the work of Hautus [73, 75]. The Sylvester equation is actually quite an old subject, and was originally introduced by Sylvester in [192].

Well-posedness was studied by Wonham in section 8.3 of [223] and by Hautus in [73]. Note that this basically still requires that if the system is perturbed then we need a new controller. In structural stability , we are looking for one controller which stabilizes a neighborhood of the given plant. This problem was studied in many variations and has been studied by Francis, Sebakhy and Wonham in [47], by Davison and Goldenberg in [36], by Desoer and Wang in [38], by Pearson, Shields and Staats in [142] and by Francis and Wonham in [48]. More recently the known results were extended by Saberi, Stoorvogel and Sannuti in the book [159].

The internal model principle studied by Wonham in [223] and by Francis and Wonham in [48] was only for the case that the to be regulated signal is equal to the measurement signal. Extensions to the general case can be found in the book [159] by Saberi, Stoorvogel and Sannuti. Note that the internal model principle is unrelated to well-posedness, structural stability (which is sometimes also referred to as robust regulation) which is sometimes alluded to in the literature. It is a basic property resulting directly from the fact that we achieve regulation.