Chapter 4

Controlled invariant subspaces

In this chapter we introduce controlled invariant subspaces (which are also called (A, B)-invariant subspaces) and the concepts of controllability subspace and stabilizability subspace. The notion of controlled invariance is of fundamental importance in many of the feedback design problems that the reader will encounter in this book. We will apply these concepts to a number of basic feedback design problems, among which the problem of disturbance decoupling by state feedback. The design problems treated in this chapter have in common that the entire state vector of the control system is assumed to be available for control purposes, and we confine ourselves to the design of static state feedback control laws. Dynamic feedback will be discussed in chapter 6.

4.1 Controlled invariance

In this section we will introduce the concept of controlled invariant subspace and prove the most important properties of these subspaces. Again consider the system

$$\dot{x}(t) = Ax(t) + Bu(t). \tag{4.1}$$

The input functions u are understood to be elements of the class $\mathbf{U}$ of admissible input functions (see section 3.1). A subspace of the state space will be called controlled invariant if it has the following property: for every initial condition in the subspace there exists an input function such that the resulting state trajectory remains in the subspace for all times. More explicitly:

Definition 4.1 *A subspace $\mathcal{V} \subset \mathcal{X}$ is called* controlled invariant *if for any $x_0 \in \mathcal{V}$ there exists an input function u such that $x_u(t, x_0) \in \mathcal{V}$ for all $t \geqslant 0$.*

It follows immediately from the definition that the sum of any number of controlled invariant subspaces is a controlled invariant subspace. In order to stress the dependence on the underlying system, we will often use the terminology (A, B)-*invariant subspace* instead of controlled invariant subspace. It is easily seen that if $F : \mathcal{X} \to \mathcal{U}$ is a linear map and $G : \mathcal{U} \to \mathcal{U}$ is an isomorphism then a given subspace $\mathcal{V}$ is (A, B)-invariant if and only if it is $(A + BF, BG)$-invariant. Stated differently: the classes of controlled invariant subspaces associated with the systems (A, B) and $(A + BF, BG)$, respectively, coincide. Sometimes this is expressed by saying that the property of controlled invariance is invariant under state feedback and isomorphism of the input space. The following theorem gives several equivalent characterizations of controlled invariance:

Theorem 4.2 *Consider the system (4.1). Let $\mathcal{V}$ be a subspace of $\mathcal{X}$. The following statements are equivalent:*

(*i*) *$\mathcal{V}$ is controlled invariant,*

(*ii*) $A\mathcal{V} \subset \mathcal{V} + \operatorname{im} B$,

(*iii*) *there exists a linear map $F : \mathcal{X} \to \mathcal{U}$ such that $(A + BF)\mathcal{V} \subset \mathcal{V}$.*

Proof : (i) $\Rightarrow$ (ii). Let $x_0 \in \mathcal{V}$ and let u be an input function such that $x_u(t, x_0) \in \mathcal{V}$ for all $t \geqslant 0$. Since $\mathcal{V}$ is a linear subspace, for all $t > 0$ we have $\frac{1}{t}(x_u(t, x_0) - x_0) \in \mathcal{V}$. Being a subspace of $\mathcal{X}$, $\mathcal{V}$ is closed in the Euclidean topology. Thus $\dot{x}(0^+) := \lim_{t\downarrow 0} \frac{1}{t}(x_u(t, x_0) - x_0) \in \mathcal{V}$. Since $Ax_0 = \dot{x}(0^+) - Bu(0^+)$ it follows that $Ax_0 \in \mathcal{V} + \operatorname{im} B$.

(ii) $\Rightarrow$ (iii). Choose a basis $q_1, \ldots, q_n$ of $\mathcal{X}$ adapted to $\mathcal{V}$. For all $1 \leqslant i \leqslant n$ there exist vectors $\bar{q}_i \in \mathcal{V}$ and $u_i \in \mathcal{U}$ such that $Aq_i = \bar{q}_i + Bu_i$. Define $F : \mathcal{X} \to \mathcal{U}$ as follows: for $1 \leqslant i \leqslant k$ define $Fq_i := -u_i$ and for $k + 1 \leqslant i \leqslant n$ let Fq_i be arbitrary vectors in $\mathcal{X}$. Then for $i = 1, \ldots, k$ we have $(A + BF)q_i = \bar{q}_i \in \mathcal{V}$ and hence $(A + BF)\mathcal{V} \subset \mathcal{V}$.

(iii) $\Rightarrow$ (i). Let $x_0 \in \mathcal{V}$. We claim that if the system is controlled by the feedback control law $u = Fx$, then the resulting trajectory remains in $\mathcal{V}$. Indeed, using this control law the trajectory $x_u(t, x_0)$ is equal to the solution of $\dot{x} = (A+BF)x,\ x(0) = x_0$. The claim then follows immediately from theorem 2.4. ■

In the above, the characterization (i) is typically an *open loop* characterization: the input functions are allowed to depend on the initial condition arbitrarily. In this vein, the characterization (iii) is called a *closed-loop* characterization: it turns out to be possible to remain within a controlled invariant subspace using a state feedback control law. As an intermediate between these two we stated (ii), a *geometric* characterization of controlled invariance.

If $\mathcal{V}$ is controlled invariant, then we will denote by $\underline{F}(\mathcal{V})$ the set of all linear maps F such that $(A + BF)\mathcal{V} \subset \mathcal{V}$. In the sequel we will often use the notation $A_F := A + BF$.

Let $\mathcal{V}$ be a controlled invariant subspace and let $F \in \underline{F}(\mathcal{V})$. Consider the equation (4.1). If we represent the control u as $u = Fx + v$, we obtain the equation

$$\dot{x}(t) = A_F x(t) + Bv(t).$$

Let $x_0 \in \mathcal{V}$. We know that if we choose $v = 0$, then the state trajectory starting in x_0 remains in $\mathcal{V}$. We now ask ourselves the question: what other control inputs v have the property that the resulting state trajectory remains in $\mathcal{V}$? We claim that the trajectory $x(t)$ starting in x_0 remains in $\mathcal{V}$ if and only if $Bv(t) \in \mathcal{V}$ for all $t \geqslant 0$. Indeed, if $x(t) \in \mathcal{V}$ for $t \geqslant 0$, then also $A_F x(t) \in \mathcal{V}$ and $\dot{x}(t) \in \mathcal{V}$ for $t \geqslant 0$. Thus $Bv(t) = \dot{x}(t) - A_F x(t) \in \mathcal{V}$ for $t \geqslant 0$. Conversely, if $Bv(t) \in \mathcal{V}$ for $t \geqslant 0$ then

$$x(t) = e^{A_F t} x_0 + \int_0^t e^{A_F (t-\tau)} Bv(\tau)\, \mathrm{d}\tau \in \mathcal{V}$$

for all $t \geqslant 0$, since $e^{A_F t} x_0 \in \mathcal{V}$ for $t \geqslant 0$. Consider the linear subspace

$$B^{-1}\mathcal{V} := \{u \in \mathcal{U} \mid Bu \in \mathcal{V}\}.$$

Then $Bv(t) \in \mathcal{V}$ is equivalent to $v(t) \in B^{-1}\mathcal{V}$. Let L be a linear map such that $\operatorname{im} L = B^{-1}\mathcal{V}$. Obviously, $v(t) \in B^{-1}\mathcal{V}$ for all $t \geqslant 0$ if and only if $v(t) = Lw(t),\ t \geqslant 0$, for some function w (compare exercise 2.2). Thus we have proven:

Theorem 4.3 *Let $\mathcal{V}$ be a controlled invariant subspace. Assume that $F \in \underline{F}(\mathcal{V})$ and let L be a linear map such that* $\operatorname{im} L = B^{-1}\mathcal{V}$. *Let $x_0 \in \mathcal{V}$ and let u be an input function. Then the state trajectory resulting from x_0 and u remains in $\mathcal{V}$ for all $t \geqslant 0$ if and only if u has the form*

$$u(t) = Fx(t) + Lw(t) \tag{4.2}$$

for some input function w.

Note that (A_F, BL) can be viewed as the restriction of the system Σ to the subspace $\mathcal{V}$. After all if we stay inside $\mathcal{V}$ then u must be of the form (4.2). Therefore, the dynamics must be of the form

$$\dot{x}(t) = A_F x(t) + BLw(t).$$

If $\mathcal{K}$ is a subspace of $\mathcal{X}$ which is not controlled invariant, then we are interested in a controlled invariant subspace contained in $\mathcal{K}$ which is as large as possible.

Definition 4.4 *Let $\mathcal{K}$ be a subspace of $\mathcal{X}$. Then we define*

$$\mathcal{V}^*(\mathcal{K}) := \{x_0 \in \mathcal{X} \mid \text{there exists an input function } u \text{ such that } x_u(t, x_0) \in \mathcal{K} \text{ for all } t \geqslant 0\}.$$

Two things follow immediately from this definition. First, it is easy to see that $\mathcal{V}^*(\mathcal{K})$ is a linear subspace of $\mathcal{X}$. Indeed, if $x_0, y_0 \in \mathcal{V}^*(\mathcal{K})$ then there are inputs u and v such that $x_u(t, x_0) \in \mathcal{K}$ and $x_v(t, y_0) \in \mathcal{K}$ for all $t \geqslant 0$. Let $\lambda, \mu \in \mathbb{R}$. Define $w(t) := \lambda u(t) + \mu v(t)$. Then $x_w(t, \lambda x_0 + \mu y_0) \in \mathcal{K}$ for all $t \geqslant 0$ (see (3.2)). Secondly, it is clear that $\mathcal{V}^*(\mathcal{K}) \subset \mathcal{K}$. In fact, we have the following result:

Theorem 4.5 *Let $\mathcal{K}$ be a subspace of $\mathcal{X}$. Then $\mathcal{V}^*(\mathcal{K})$ is the largest controlled invariant subspace contained in $\mathcal{K}$, i.e.*

(*i*) *$\mathcal{V}^*(\mathcal{K})$ is a controlled invariant subspace,*

(*ii*) *$\mathcal{V}^*(\mathcal{K}) \subset \mathcal{K}$,*

(*iii*) *if $\mathcal{V} \subset \mathcal{K}$ is a controlled invariant subspace then $\mathcal{V} \subset \mathcal{V}^*(\mathcal{K})$.*

Proof : We first show that $\mathcal{V}^*(\mathcal{K})$ is controlled invariant. Assume $x_0 \in \mathcal{V}^*(\mathcal{K})$. There is an input u such that $x_u(t, x_0) \in \mathcal{K}$ for all $t \geqslant 0$. We claim that, in fact, $x_u(t, x_0) \in \mathcal{V}^*(\mathcal{K})$ for all $t \geqslant 0$. To show this, take a fixed but arbitrary $t_1 \geqslant 0$. Let $x_1 := x_u(t_1, x_0)$. It will be shown that $x_1 \in \mathcal{V}^*(\mathcal{K})$. Indeed, if we define $v(t) := u(t_1 + t)$ $(t \geqslant 0)$, then using (3.2) we have $x_v(t, x_1) = x_u(t + t_1, x_0) \in \mathcal{K}$ for all $t \geqslant 0$. This proves that $x_1 = x_u(t_1, x_0)$ lies in $\mathcal{V}^*(\mathcal{K})$. Since t_1 was arbitrary, $x_u(t, x_0) \in \mathcal{V}^*(\mathcal{K})$ for *all* $t \geqslant 0$ and hence $\mathcal{V}^*(\mathcal{K})$ is controlled invariant.

Next, we show that $\mathcal{V}^*(\mathcal{K})$ is the largest controlled invariant subspace in $\mathcal{K}$. Let $\mathcal{V} \subset \mathcal{K}$ be controlled invariant. Let $x_0 \in \mathcal{V}$. There is an input u such that $x_u(t, x_0) \in \mathcal{V}$ for all $t \geqslant 0$. Consequently, $x_u(t, x_0) \in \mathcal{K}$ for all $t \geqslant 0$ and hence $x_0 \in \mathcal{V}^*(\mathcal{K})$. This completes the proof. ■

In order to display the dependence on the underlying system we will sometimes denote $\mathcal{V}^*(\mathcal{K})$ by $\mathcal{V}^*(\mathcal{K}, A, B)$.

4.2 Disturbance decoupling

Consider the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Ed(t), \\ z(t) &= Hx(t). \end{aligned} \tag{4.3}$$

In the differential equation above, d represents an unknown disturbance which is assumed to be an element of a given function space. For a given initial condition x_0 and disturbance d, the output of the system is given by

$$z(t) = He^{At}x_0 + \int_0^t T(t-\tau)d(\tau)\,\mathrm{d}\tau. \tag{4.4}$$

Here $T(t) := He^{At}E$ is the impulse response between the disturbance and the output. The system (4.3) will be called *disturbance decoupled* if $T = 0$ or, equivalently, if the transfer function $G(s) = H(Is - A)^{-1}E$ is equal to zero. If this is the case then for any given initial condition x_0 the output is equal to $z(t) = He^{At}x_0$ for all disturbances d. This means that in a system which is disturbance decoupled, the output does not depend on the disturbance. The following theorem will play an important role in the sequel:

Theorem 4.6 *The system (4.3) is disturbance decoupled if and only if there exists an A-invariant subspace $\mathcal{V}$ such that* $\operatorname{im} E \subset \mathcal{V} \subset \ker H$.

Proof : ($\Rightarrow$) If $T = 0$ then also all its time derivatives $T^{(k)}$ are identically equal to 0. Thus, $T^{(k)}(t) = HA^k e^{At}E = 0$ for all t. By taking $t = 0$ this yields $HA^kE = 0$, $k = 0, 1, 2, \ldots$ Define $\mathcal{V} := \operatorname{im}\begin{pmatrix} E & AE & \cdots & A^{n-1}E \end{pmatrix}$. Then $\mathcal{V} \subset \ker H$. By corollary 3.3, $\mathcal{V}$ is equal to $\langle A \mid \operatorname{im} E \rangle$, the smallest A-invariant subspace that contains $\operatorname{im} E$.

($\Leftarrow$) If $\operatorname{im} E \subset \mathcal{V}$ and $\mathcal{V}$ is A-invariant, then also $\operatorname{im} A^kE \subset \mathcal{V}$ for $k = 0, 1, 2, \ldots$. Since we know that $\mathcal{V} \subset \ker H$ this yields $HA^kE = 0$ for all k. Thus $T(t) = \sum_{k=0}^{\infty}(t^k/k!)HA^kE = 0$ for all t. It follows that the system is disturbance decoupled. ∎

If the system (4.3) is not disturbance decoupled, then one can try to *make* it disturbance decoupled. In order to do this one needs the possibility to change the system's dynamics by using a control input. This possibility is modelled by adding a control term to the right hand side of the original differential equation in (4.3). Thus we consider the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ed(t), \\ z(t) &= Hx(t). \end{aligned} \tag{4.5}$$

In this description, the variable u represents a control input (see also section 2.8). Let $F : \mathcal{X} \to \mathcal{U}$ be a linear map. If in (4.5) we substitute $u(t) = Fx(t)$, the system's equations change into

$$\begin{aligned} \dot{x}(t) &= (A + BF)x(t) + Ed(t), \\ z(t) &= Hx(t), \end{aligned} \tag{4.6}$$

the closed-loop system obtained from the state feedback control law $u = Fx$. The impulse response matrix of (4.6) is called the *closed-loop impulse response* and is equal to

$$T_F(t) := He^{(A+BF)t}E.$$

The corresponding transfer function $G_F(s) := H(Is - A - BF)^{-1}E$ is called the *closed-loop transfer function*. The problem of disturbance decoupling by state feedback is to find a linear map $F : \mathcal{X} \to \mathcal{U}$ such that the closed-loop system (4.6) is disturbance decoupled:

Definition 4.7 *Consider the system (4.3).* The problem of disturbance decoupling by state feedback, DDP, *is to find a linear map* $F : \mathcal{X} \to \mathcal{U}$ *such that* $T_F = 0$ *(or, equivalently, such that* $G_F = 0$*).*

The following result establishes the connection between the concept of controlled invariance and the problem of disturbance decoupling.

Theorem 4.8 *There exists a linear map* $F : \mathcal{X} \to \mathcal{U}$ *such that* $T_F = 0$ *if and only if there exists a controlled invariant subspace* $\mathcal{V}$ *such that* $\operatorname{im} E \subset \mathcal{V} \subset \ker H$.

Proof : ($\Rightarrow$) If $T_F = 0$ then (4.6) is disturbance decoupled. By theorem 4.6 there is an $(A + BF)$-invariant subspace $\mathcal{V}$ such that $\operatorname{im} E \subset \mathcal{V} \subset \ker H$. By theorem 4.2, $\mathcal{V}$ is controlled invariant.

($\Leftarrow$) Let $\mathcal{V}$ be a controlled invariant subspace such that $\operatorname{im} E \subset \mathcal{V} \subset \ker H$. By theorem 4.2 there exists a linear map $F : \mathcal{X} \to \mathcal{U}$ such that $\mathcal{V}$ is $(A+BF)$-invariant. It then follows from theorem 4.6 that the system (4.6) is disturbance decoupled. ■

Corollary 4.9 *There exists a linear map* $F : \mathcal{X} \to \mathcal{U}$ *such that* $T_F = 0$ *if and only if*

$$\operatorname{im} E \subset \mathcal{V}^*(\ker H). \tag{4.7}$$

Formula (4.7) provides a very compact necessary and sufficient condition for the existence of a state feedback control law that achieves disturbance decoupling. However, in order to be able to check this condition for an actual system, we would like to have an algorithm. In the next section we will describe an algorithm that, starting from a system (4.5), calculates the associated subspace $\mathcal{V}^*(\ker H)$.

4.3 The invariant subspace algorithm

In this section we give an algorithm to compute the subspace $\mathcal{V}^*(\mathcal{K})$. Consider the system (A, B) and let $\mathcal{K}$ be a subspace of the state space $\mathcal{X}$. The algorithm we give is most easily understood if one thinks in terms of the discrete-time system

$$x_{t+1} = Ax_t + Bu_t, \quad t = 0, 1, 2, \ldots. \tag{4.8}$$

Given an input sequence $u = (u_0, u_1, u_2, \ldots)$ and an initial condition x_0, the resulting discrete-time state trajectory is denoted by $x = (x_0, x_1, x_2, \ldots)$. The discrete-time analogue $\mathcal{V}_d^*(\mathcal{K})$ of the subspace $\mathcal{V}^*(\mathcal{K})$ defined by definition 4.4 is obviously the subspace of all $x_0 \in \mathcal{X}$ for which there exists an input sequence u such that all terms of the resulting state trajectory lie in $\mathcal{K}$:

$$\mathcal{V}_d^*(\mathcal{K}) := \{x_0 \in \mathcal{X} \mid \text{ there is an input sequence } u \text{ such that } x_t \in \mathcal{K} \text{ for } t = 0, 1, 2, \ldots\}.$$

Define a sequence of subspaces $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \ldots$ by

$$\mathcal{V}_t := \{x_0 \in \mathcal{X} \mid \text{there is an input sequence } u \text{ such that } x_0, x_1, x_2, \ldots, x_t \in \mathcal{K}\}.$$

Thus, $\mathcal{V}_t$ consists of those points in which a state trajectory starts for which the first $t+1$ terms lie in $\mathcal{K}$. It is easily verified that $\mathcal{V}_t$ is indeed a subspace, that $\mathcal{V}_0 = \mathcal{K}$ and that $\mathcal{V}_0 \supset \mathcal{V}_1 \supset \mathcal{V}_2 \supset \cdots$. It turns out to be possible to derive a recurrence relation for $\mathcal{V}_t$. Indeed, $x_0 \in \mathcal{V}_{t+1}$ if and only if $x_0 \in \mathcal{K}$ and there exists $u_0 \in \mathcal{U}$ such that $Ax_0 + Bu_0 \in \mathcal{V}_t$. Hence, $x_0 \in \mathcal{V}_{t+1}$ if and only if $x_0 \in \mathcal{K}$ and $Ax_0 \in \mathcal{V}_t + \operatorname{im} B$ or, equivalently, $x_0 \in A^{-1}(\mathcal{V}_t + \operatorname{im} B)$. It follows that

$$\mathcal{V}_0 = \mathcal{K}, \;\; \mathcal{V}_{t+1} = \mathcal{K} \cap A^{-1}(\mathcal{V}_t + \operatorname{im} B). \tag{4.9}$$

From this recurrence relation it follows immediately that if $\mathcal{V}_k = \mathcal{V}_{k+1}$ for some k, then $\mathcal{V}_k = \mathcal{V}_t$ for all $t \geqslant k$. Now, recall that $\mathcal{V}_0 \supset \mathcal{V}_1 \supset \mathcal{V}_2 \supset \cdots$. If we have strict inclusion, the dimension must decrease by at least one. Hence the inclusion chain must have the form

$$\mathcal{V}_0 \supset \mathcal{V}_1 \supset \cdots \supset \mathcal{V}_k = \mathcal{V}_{k+1} = \mathcal{V}_{k+2} = \cdots$$

for some integer $k \leqslant \dim \mathcal{K}$ $(\leqslant n-1)$. In the above formula, $\supset$ stands for *strict* inclusion. We claim that $\mathcal{V}_d^*(\mathcal{K}) = \mathcal{V}_k$. Indeed, on the one hand it follows immediately from the definition that $\mathcal{V}_d^*(\mathcal{K}) \subset \mathcal{V}_t$ for all t. Conversely, assume $x_0 \in \mathcal{V}_k$. We want to construct an input sequence $(u_0, u_1, \ldots)$ such that the corresponding state trajectory $x = (x_0, x_1, \ldots)$ lies in $\mathcal{K}$. Since $x_0 \in \mathcal{V}_k = \mathcal{V}_{k+1}$, there is u_0 such that $x_1 = Ax_0 + Bu_0 \in \mathcal{V}_k$. Thus, in particular we have $x_0, x_1 \in \mathcal{K}$. We now proceed inductively. Assume $u_0, u_1, \ldots, u_{s-1}$ have been found such that $x_0, x_1, \ldots, x_{s-1} \in \mathcal{K}$, while $x_s \in \mathcal{V}_k$. Again using $\mathcal{V}_k = \mathcal{V}_{k+1}$ we can find u_s such that $x_{s+1} = Ax_s + Bu_s \in \mathcal{V}_k$. This proves our claim.

The above is meant to provide some intuitive background for the introduction of the recurrence relation (4.9). This recurrence relation will henceforth be called *the invariant subspace algorithm*, ISA. Of course, we still have to show its relevance in continuous-time systems. In the following result we will collect the properties of the sequence $\{\mathcal{V}_t\}$ we established above and prove that it can be used to calculate the largest controlled invariant subspace contained in $\mathcal{K}$ for the continuous-time system (4.1).

Theorem 4.10 *Consider the system (4.1). Let $\mathcal{K}$ be a subspace of $\mathcal{X}$. Let $\mathcal{V}_t$, $t = 0, 1, 2, \ldots$, be defined by the algorithm (4.9). Then we have*

(*i*) $\mathcal{V}_0 \supset \mathcal{V}_1 \supset \mathcal{V}_2 \supset \cdots$,

(*ii*) *there exists* $k \leqslant \dim \mathcal{K}$ *such that* $\mathcal{V}_k = \mathcal{V}_{k+1}$,

(*iii*) *if* $\mathcal{V}_k = \mathcal{V}_{k+1}$ *then* $\mathcal{V}_k = \mathcal{V}_t$ *for all* $t \geqslant k$,

(*iv*) *if* $\mathcal{V}_k = \mathcal{V}_{k+1}$ *then* $\mathcal{V}^*(\mathcal{K}) = \mathcal{V}_k$.

Proof : The statements (i), (ii) and (iii) were proven above. To prove (iv), note that $\mathcal{V}_k \subset \mathcal{K}$. Moreover, it follows immediately from (4.9) that

$$A\mathcal{V}_k = A\mathcal{V}_{k+1} \subset \mathcal{V}_k + \operatorname{im} B.$$

Hence $\mathcal{V}_k$ is a controlled invariant subspace contained in $\mathcal{K}$ and therefore contained in $\mathcal{V}^*$. To prove the converse inclusion, we show that in fact $\mathcal{V}^* \subset \mathcal{V}_t$ for all t. Obviously this is true for $t = 0$. Assume $\mathcal{V}^* \subset \mathcal{V}_{t-1}$. Since $\mathcal{V}^*$ is controlled invariant we have $A\mathcal{V}^* \subset \mathcal{V}^* + \operatorname{im} B$. Thus $A\mathcal{V}^* \subset \mathcal{V}_{t-1} + \operatorname{im} B$ and hence $\mathcal{V}^* \subset A^{-1}(\mathcal{V}_{t-1} + \operatorname{im} B)$. Finally, $\mathcal{V}^* \subset \mathcal{K}$ so we conclude that $\mathcal{V}^* \subset \mathcal{V}_t$. ■

Of course item (iv) is crucial because it tells us that there exists a *finite* algorithm to compute $\mathcal{V}^*$. As a matter of fact it is easily seen that we need at most n steps where n is the dimension of the state space.

4.4 Controllability subspaces

Consider the system (4.1). If a subspace of the state space has the property that every point in that subspace can be steered to the origin in finite time without leaving the subspace, it is called a controllability subspace.

Definition 4.11 *A subspace $\mathcal{R} \subset \mathcal{X}$ is called a* controllability subspace *if for every $x_0 \in \mathcal{R}$ there exists $T > 0$ and an input function u such that $x_u(t, x_0) \in \mathcal{R}$ for $0 \leqslant t \leqslant T$ and $x_u(T, x_0) = 0$.*

It is immediately clear from this definition that every controllability subspace is controlled invariant. Indeed, if one chooses the control input to be equal to zero for $t > T$, the state trajectory also remains zero and hence does not leave $\mathcal{R}$. As was the case with controlled invariant subspaces, it can be shown that the sum of any (finite or infinite) number of controllability subspaces is a controllability subspace. Also, the class of all controllability subspaces associated with a given system is invariant under state feedback and isomorphisms of the input space. That is, if $\mathcal{R}$ is a controllability subspace with respect to (A, B), it is a controllability subspace with respect to $(A + BF, BG)$ for all linear maps $F : \mathcal{X} \to \mathcal{U}$ and isomorphisms G of $\mathcal{U}$.

We can give the following characterization of controllability subspaces:

Theorem 4.12 *A subspace $\mathcal{R} \subset \mathcal{X}$ is a controllability subspace if and only if there exist linear maps F and L such that*

$$\mathcal{R} = \langle A + BF \mid \operatorname{im} BL\rangle. \tag{4.10}$$

Proof : ($\Rightarrow$) Let $F \in \underline{F}(\mathcal{R})$ and L be a linear map such that $\operatorname{im} L = B^{-1}\mathcal{R}$. We claim that (4.10) holds. Let $x_0 \in \mathcal{R}$. There is $T > 0$ and an input u such that $x_u(t, x_0) \in \mathcal{R}$ for all $t \geqslant 0$ and $x_u(T, x_0) = 0$. By theorem 4.3 there exists w such that $u(t) = Fx_u(t, x_0) + Lw(t)$. Hence, $x_u(t, x_0)$ is a state trajectory of the system $\dot{x}(t) = A_F x(t) + BLw(t)$ with state space $\mathcal{R}$. Along this trajectory, x_0 is steered to 0 at time $t = T$. Since this is possible for all $x_0 \in \mathcal{R}$, it follows that the latter system is null-controllable. Consequently, it is reachable so $\mathcal{R} = \langle A_F \mid \operatorname{im} BL \rangle$ (see section 3.2).

($\Leftarrow$) Assume that (4.10) holds. Then we have $A_F \mathcal{R} \subset \mathcal{R}$ and $\operatorname{im} BL \subset \mathcal{R}$. Thus $\dot{x}(t) = A_F x(t) + BLw(t)$ defines a system with state space $\mathcal{R}$. By corollary 3.4, this system is controllable. Hence every point in $\mathcal{R}$ can be controlled to the origin in finite time while remaining in $\mathcal{R}$. ■

It follows from the proof of the above theorem that if $\mathcal{R}$ is a controllability subspace then for the maps F and L in the representation (4.10) we can take any $F \in \underline{F}(\mathcal{R})$ and any map L such that $\operatorname{im} L = B^{-1}\mathcal{R}$. Since the latter equality implies $\operatorname{im} BL = \operatorname{im} B \cap \mathcal{R}$ we obtain the following:

Corollary 4.13 *Let $\mathcal{R}$ be a controllability subspace. Then for any $F \in \underline{F}(\mathcal{R})$ we have*

$$\mathcal{R} = \langle A + BF \mid \operatorname{im} B \cap \mathcal{R} \rangle.$$

If $\mathcal{K}$ is a subspace of the state space $\mathcal{X}$ then we are interested in the largest controllability subspace that is contained in $\mathcal{K}$ (see also definition 4.4).

Definition 4.14 *Let $\mathcal{K}$ be a subspace of $\mathcal{X}$. Then we define*

$$\mathcal{R}^*(\mathcal{K}) := \{x_0 \in \mathcal{X} \mid \text{there exists an input function } u \text{ and } T > 0 \text{ such that } x_u(t, x_0) \in \mathcal{K} \text{ for all } 0 \leqslant t \leqslant T \text{ and } x_u(T, x_0) = 0\}.$$

Clearly, $\mathcal{R}^*(\mathcal{K})$ is contained in $\mathcal{K}$. In fact, we have

Theorem 4.15 *Let $\mathcal{K}$ be a subspace of $\mathcal{X}$. Then $\mathcal{R}^*(\mathcal{K})$ is the largest controllability subspace contained in $\mathcal{K}$, i.e.*

(*i*) *$\mathcal{R}^*(\mathcal{K})$ is a controllability subspace,*

(*ii*) *$\mathcal{R}^*(\mathcal{K}) \subset \mathcal{K}$,*

(*iii*) *if $\mathcal{R} \subset \mathcal{K}$ is a controllability subspace then $\mathcal{R} \subset \mathcal{R}^*(\mathcal{K})$.*

Proof : We first show that $\mathcal{R}^*(\mathcal{K})$ is a subspace. Let $x_0, y_0 \in \mathcal{R}^*(\mathcal{K})$ and $\lambda, \mu \in \mathbb{R}$. There exist controls u, v and numbers $T_1, T_2 > 0$ such that $x_u(T_1, x_0) = 0$, $x_v(T_2, y_0) = 0$, $x_u(t, x_0) \in \mathcal{K}$ for all $0 \leqslant t \leqslant T_1$ and $x_v(t, y_0) \in \mathcal{K}$ for all $0 \leqslant t \leqslant T_2$. Without loss of generality, assume that $T_1 \leqslant T_2$. Define a new control function $\tilde{u}$ by $\tilde{u}(t) = u(t)$ $(0 \leqslant t \leqslant T_1)$ and $\tilde{u}(t) = 0$ $(t > T_1)$. Then $x_{\tilde{u}}(T_2, x_0) = 0$ and $x_{\tilde{u}}(t, x_0)(t) \in \mathcal{K}$ for all $0 \leqslant t \leqslant T_2$. Define now $w(t) := \lambda \tilde{u}(t) + \mu v(t)$. Then $x_w(T_2, \lambda x_0 + \mu y_0) = 0$ and $x_w(t, \lambda x_0 + \mu y_0) \in \mathcal{K}$ for all $0 \leqslant t \leqslant T_2$.

Next, we prove that $\mathcal{R}^*(\mathcal{K})$ is a controllability subspace. Let $x_0 \in \mathcal{R}^*(\mathcal{K})$. There is a control input u and a number $T > 0$ such that $x_u(T, x_0) = 0$ and $x_u(t, x_0) \in \mathcal{K}$ for all $0 \leqslant t \leqslant T$. We contend that, in fact, $x_u(t, x_0) \in \mathcal{R}^*(\mathcal{K})$. To prove this, take a fixed but arbitrary $t_1 < T$. Let $x_1 := x_u(t_1, x_0)$. Define a new input function v by $v(t) := u(t + t_1)$ $(t \geqslant 0)$. Then $x_v(t, x_1) = x_u(t_1 + t, x_0) \in \mathcal{K}$ for all $0 \leqslant t \leqslant T - t_1$ and $x_v(T - t_1, x_1) = x_u(T, x_0) = 0$. Consequently, x_1 can be controlled to the origin in finite time while remaining in $\mathcal{K}$ and hence $x_1 \in \mathcal{R}^*(\mathcal{K})$. Since t_1 was arbitrary we find that $x_u(t, x_0) \in \mathcal{R}^*(\mathcal{K})$ for all $0 \leqslant t \leqslant T$. Finally, the fact that $R^*(\mathcal{K})$ is the largest controllability subspace in $\mathcal{K}$ is proven completely similarly as the corresponding part of theorem 4.5. ■

Sometimes, we will denote $\mathcal{R}^*(\mathcal{K})$ by $\mathcal{R}^*(\mathcal{K}, A, B)$. Starting with a subspace $\mathcal{K}$ of the state space, we have now defined $\mathcal{V}^*(\mathcal{K})$ (see definition 4.4) and $\mathcal{R}^*(\mathcal{K})$. Since $\mathcal{R}^*(\mathcal{K})$ is controlled invariant and contained in $\mathcal{K}$, it must be contained in the largest controlled invariant subspace in $\mathcal{K}$. Thus

$$\mathcal{R}^*(\mathcal{K}) \subset \mathcal{V}^*(\mathcal{K}) \subset \mathcal{K}. \tag{4.11}$$

More specifically, we have $\mathcal{R}^*(\mathcal{V}^*(\mathcal{K})) = \mathcal{R}^*(\mathcal{K})$. In the following, whenever this is convenient, we denote $\mathcal{R}^*(\mathcal{K})$ and $\mathcal{V}^*(\mathcal{K})$ by $\mathcal{R}^*$ and $\mathcal{V}^*$, respectively.

Lemma 4.16 *Let $\mathcal{K}$ be a subspace of $\mathcal{X}$. Then* $\operatorname{im} B \cap \mathcal{V}^*(\mathcal{K}) \subset \mathcal{R}^*(\mathcal{K})$.

Proof : Let L be a linear map such that $\operatorname{im} L = B^{-1}\mathcal{V}^*$. Then $\operatorname{im} BL = \operatorname{im} B \cap \mathcal{V}^*$. Choose $F \in \underline{F}(\mathcal{V}^*)$. Then we have

$$\operatorname{im} B \cap \mathcal{V}^* \subset \langle A_F \mid \operatorname{im} BL \rangle \subset \mathcal{V}^* \subset \mathcal{K}.$$

Since $\langle A_F \mid \operatorname{im} BL \rangle$ is a controllability subspace (see theorem 4.12) it must be contained in $\mathcal{R}^*(\mathcal{K})$. This proves the lemma. ■

The above lemma will be used to prove the following, stronger, result:

Theorem 4.17 *Let $\mathcal{K}$ be a subspace of $\mathcal{X}$. Then $\underline{F}(\mathcal{V}^*) \subset \underline{F}(\mathcal{R}^*)$ and*

$$\mathcal{R}^* = \langle A + BF \mid \operatorname{im} B \cap \mathcal{V}^* \rangle \tag{4.12}$$

for all $F \in \underline{F}(\mathcal{V}^*)$.

In the above, $\underline{F}(\mathcal{R}^*)$ denotes the set of all linear maps F with the property that $\mathcal{R}^*$ is $A + BF$ invariant. This is consistent with our earlier notation $\underline{F}(\mathcal{V})$ with respect to the controlled invariant subspace $\mathcal{V}$, since every controllability subspace is controlled invariant.

Proof : Let $F \in \underline{F}(\mathcal{V}^*)$. Since $\mathcal{R}^* \subset \mathcal{V}^*$ we have that $A_F\mathcal{R}^* \subset \mathcal{V}^*$. On the other hand, since $\mathcal{R}^*$ is controlled invariant, $A_F\mathcal{R}^* \subset \mathcal{R}^* + \operatorname{im} B$. Thus we have

$$A_F\mathcal{R}^* \subset (\operatorname{im} B + \mathcal{R}^*) \cap \mathcal{V}^* = (\operatorname{im} B \cap \mathcal{V}^*) + \mathcal{R}^* \subset \mathcal{R}^*.$$

where we used the modular rule for the equality in the middle. This shows that $F \in \underline{F}(\mathcal{R}^*)$. Next, by corollary 4.13, $\mathcal{R}^* = \langle A_F \mid \operatorname{im} B \cap \mathcal{R}^*\rangle$. Moreover, it follows from lemma 4.16 that $\operatorname{im} B \cap \mathcal{R}^* = \operatorname{im} B \cap \mathcal{V}^*$. This completes the proof of the theorem. ■

The above theorem has the following interpretation. By taking $F \in \underline{F}(\mathcal{V}^*)$ and a linear map L such that $\operatorname{im} L = B^{-1}\mathcal{V}^*$ we obtain a new system

$$\dot{x}(t) = (A + BF)x(t) + BLw(t)$$

with state space $\mathcal{V}^*$. This system can be considered as being obtained from the original system by restricting the trajectories to the subspace $\mathcal{V}^*$ and by restricting the input functions to take their values in $B^{-1}\mathcal{V}^*$. Since $\operatorname{im} BL = \operatorname{im} B \cap \mathcal{V}^*$, (4.12) expresses the fact that $\mathcal{R}^*$ is just the reachable subspace of this restricted system.

If $\mathcal{V}$ is a controlled invariant subspace then of course $\mathcal{V} = \mathcal{V}^*(\mathcal{V})$. Let $\mathcal{R} := \mathcal{R}^*(\mathcal{V})$, the largest controllability subspace contained in $\mathcal{V}$. Then theorem 4.17 says that if $F \in \underline{F}(\mathcal{V})$ and L is a linear map such that $\operatorname{im} L = B^{-1}\mathcal{V}$ then

$$\mathcal{R} = \langle A + BF \mid \operatorname{im} BL\rangle. \tag{4.13}$$

Finally we note that it follows from theorem 4.17 that $\mathcal{R}^*(\mathcal{X})$, the largest controllability subspace of the system (4.1), is equal to the reachable subspace $\langle A \mid \operatorname{im} B\rangle$. Indeed, the state space $\mathcal{X}$ itself is of course a controlled invariant subspace so $\mathcal{V}^*(\mathcal{X}) = \mathcal{X}$ and $\underline{F}(\mathcal{X}) = \{F : \mathcal{X} \to \mathcal{U} \mid F \text{ is linear}\}$. It also follows from this that every controllability subspace $\mathcal{V}$ is contained in $\langle A \mid \operatorname{im} B\rangle$.

4.5 Pole placement under invariance constraints

In section 3.10 we have discussed to what extent one can assign the spectrum of the system map using state feedback. In section 4.1 we introduced the class of controlled invariant subspaces and showed that these are characterized by the property that they can be made invariant by state feedback. In the present section we will combine these two issues and ask ourselves the question: how much freedom is left in the assignment of the spectrum of the system map if it is required that a given controlled invariant subspace should be made invariant? More concretely: given a controlled invariant subspace $\mathcal{V}$, what freedom do we have in the assignment of the spectrum of $A + BF$ if we restrict ourselves to $F \in \underline{F}(\mathcal{V})$. The following result gives a complete solution.

Theorem 4.18 *Consider the system (4.1). Let $\mathcal{V}$ be a controlled invariant subspace. Let $\mathcal{R} := \mathcal{R}^*(\mathcal{V})$ be the largest controllability subspace contained in $\mathcal{V}$. Let $\mathcal{S} := \mathcal{V} + \langle A \mid \operatorname{im} B\rangle$. Then we have* (*i*) $\underline{F}(\mathcal{V}) \subset \underline{F}(\mathcal{R}) \cap \underline{F}(\mathcal{S})$.

(*ii*) *Given any pair of real monic polynomials* (p_1, p_2) *with* $\deg p_1 = \dim \mathcal{R}$ *and* $\deg p_2 = \dim \mathcal{S}/\mathcal{V}$ *there exists* $F \in \underline{F}(\mathcal{V})$ *such that the characteristic polynomials of* $A_F \mid \mathcal{R}$ *and* $A_F \mid \mathcal{S}/\mathcal{V}$ *equal* p_1 *and* p_2*, respectively.*

(*iii*) *The map* $A_F \mid \mathcal{V}/\mathcal{R}$ *is independent of* F *for* $F \in \underline{F}(\mathcal{V})$*. The map* $A_F \mid \mathcal{X}/\mathcal{S}$ *is equal to* $A \mid \mathcal{X}/\mathcal{S}$ *for all* F*.*

The results concerning the freedom of spectral assignability under the constraint that a given controlled invariant subspace should be made invariant is depicted in the lattice diagram in Figure 4.1. Before we establish a proof of this theorem, let us

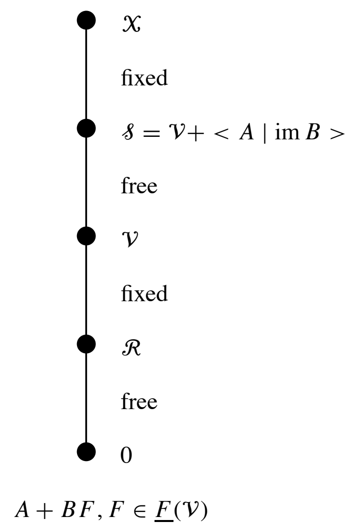

Figure 4.1

make some remarks. The theorem states that if a feedback map F makes $\mathcal{V}$ invariant under $A + BF$, then it must do the same with $\mathcal{R}$ and $\mathcal{S}$ (see also theorem 4.17). The subspaces $\mathcal{R}$, $\mathcal{V}$ and $\mathcal{S}$ form a chain, that is, they are related by the inclusion relation

$$\mathcal{R} \subset \mathcal{V} \subset \mathcal{S}. \tag{4.14}$$

In order to appreciate the content of theorem 4.18 it is useful to see what it says in terms of partitioned matrices. Choose a basis of the state space $\mathcal{X}$ adapted to the chain (4.14). Accordingly, we can split

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ 0 \end{pmatrix}.$$

Here, the zero blocks appear due to the facts that the subspace $\mathcal{S}$ is A-invariant and contains the subspace im B. A map $F = \begin{pmatrix} F_1 & F_2 & F_3 & F_4 \end{pmatrix}$ is an element of $\underline{F}(\mathcal{V})$ if and only if

$$A_{31} + B_3 F_1 = 0 \text{ and } A_{32} + B_3 F_2 = 0. \tag{4.15}$$

If we restrict ourselves to maps F satisfying (4.15) then automatically $A_{21} + B_2 F_1 = 0$ (see theorem 4.17). Theorem 4.18 asserts that, under the restriction that F_1 should satisfy (4.15), the spectrum of $A_{11} + B_1 F_1$ is freely assignable. Also, the eigenvalues of $A_{33} + B_3 F_3$ can be placed arbitrarily by appropriate choice of F_3. Finally, the theorem states that, under the restriction (4.15) on F_2, the map $A_{22} + B_2 F_2$ is fixed. More specifically, if F_2^1 and F_2^2 satisfy $A_{32} + B_3 F_2^i = 0$ $(i = 1, 2)$ then we have $A_{22} + B_2 F_2^1 = A_{22} + B_2 F_2^2$ and, a fortiori, also $\sigma(A_{22} + B_2 F_2^1) = \sigma(A_{22} + B_2 F_2^2)$. The block A_{44} is not affected by any feedback map F and consequently also $\sigma(A_{44})$ is fixed.

As a consequence of the above theorem, given a pair of real monic polynomials (p_1, p_2) with $\deg p_1 = \dim \mathcal{R}$ and $\deg p_2 = \dim \mathcal{S}/\mathcal{V}$, we can find a linear map $(F_1 \; F_2 \; F_3 \; F_4) = F \in \underline{F}(\mathcal{V})$ such that the characteristic polynomial of $A + BF$ becomes equal to the product $p_1 \cdot q \cdot p_2 \cdot r$. Here q is equal to the characteristic polynomial of $A_{22} + B_2 F_2$ which, as noted before, is the same for all maps F_2 such that $F \in \underline{F}(\mathcal{V})$. The polynomial r is equal to the characteristic polynomial of A_{44}.

In the proof of theorem 4.18 the following lemma will be useful.

Lemma 4.19 *Let $\mathcal{V}$ be a controlled invariant subspace and let $F_0 \in \underline{F}(\mathcal{V})$. Let $F : \mathcal{X} \to \mathcal{U}$ be a linear map. Then $F \in \underline{F}(\mathcal{V})$ if and only if $(F_0 - F)\mathcal{V} \subset B^{-1}\mathcal{V}$.*

Proof : ($\Rightarrow$) Let $x_0 \in \mathcal{V}$. Then $(A + BF_0)x_0 \in \mathcal{V}$ and $(A + BF)x_0 \in \mathcal{V}$. Hence $B(F_0 - F)x_0 \in \mathcal{V}$.

($\Leftarrow$) Let $x_0 \in \mathcal{V}$. Then $B(F_0 - F)x_0 \in \mathcal{V}$. Also $(A + BF_0)x_0 \in \mathcal{V}$. It follows that also $(A + BF)x_0 = (A + BF_0)x_0 - B(F_0 - F)x_0 \in \mathcal{V}$. ■

We will now give a proof of theorem 4.18.

Proof of theorem 4.18: : (i) The fact that $\underline{F}(\mathcal{V}) \subset \underline{F}(\mathcal{R})$ was already proven in theorem 4.17. The subspace $\mathcal{S}$ is invariant under $A + BF$ for any F (see also exercise 4.2).

(ii) Let (p_1, p_2) be a pair of polynomials as in the statement of the theorem. We choose any $F_0 \in \underline{F}(\mathcal{V})$ and $L : \mathbb{R}^k \to \mathcal{U}$ with $\operatorname{im} L = B^{-1}\mathcal{V}$, where $k := \dim B^{-1}\mathcal{V}$. Then, according to (4.13), we have

$$\mathcal{R} = \langle A + BF_0 \mid \operatorname{im} BL \rangle.$$

Define $A_0 := (A + BF_0) \mid \mathcal{R}$ and $B_0 := BL$. Then the system (A_0, B_0) is controllable (see corollary 3.4) and hence, by theorem 3.29, there exists a map $F_1 : \mathcal{R} \to \mathbb{R}^k$

such that $A_0 + B_0F_1$ has characteristic polynomial p_1. Extend F_1 to a linear map from $\mathcal{X} \to \mathbb{R}^k$. Define $F_2 := F_0 + LF_1$. Since $F_2 - F_0 = LF_1$ and $\operatorname{im} L = B^{-1}\mathcal{V}$, it follows from lemma 4.19 that $\mathcal{V}$ is invariant under $A + BF_2$. Also $\mathcal{S}$ is $(A + BF_2)$-invariant. Let $\Pi : \mathcal{S} \to \mathcal{S}/\mathcal{V}$ be the canonical projection (see section 2.4). Define $A_2 := (A+BF_2) \mid \mathcal{S}/\mathcal{V}$ and let $B_2 : \mathcal{U} \to \mathcal{S}/\mathcal{V}$ be defined by $B_2 := \Pi B$. We claim that the system (A_2, B_2) is controllable. We will show that $\mathcal{S}/\mathcal{V} = \langle A_2 \mid \operatorname{im} B_2\rangle$. Let $\bar{x} \in \mathcal{S}/\mathcal{V}$, say $\bar{x} = \Pi x$ with $x \in \mathcal{S}$. Then x can be written as $x = x_1 + x_2$ with $x_1 \in \langle A \mid \operatorname{im} B\rangle = \langle A + BF_2 \mid \operatorname{im} B\rangle$ and $x_2 \in \mathcal{V}$. Since $\mathcal{V} = \ker \Pi$ we have that in fact $\bar{x} = \Pi x_1$. There are $u_0, \dots, u_{n-1} \in \mathcal{U}$ such that $x_1 = \sum_i (A + BF_2)^i Bu_i$. Thus

$$\begin{aligned}\bar{x} = \Pi x_1 &= \sum_i \Pi(A + BF_2)^i Bu_i \\ &= \sum_i A_2^i B_2 u_i \in \langle A_2 \mid \operatorname{im} B_2\rangle.\end{aligned}$$

Here, we have used the fact that $\Pi(A + BF_2) = A_2\Pi$ and $B_2 = \Pi B$. This proves our claim. Now, by theorem 3.29 there exists a map $\bar{F}_3 : \mathcal{S}/\mathcal{V} \to \mathcal{U}$ such that the characteristic polynomial of $A_2 + B_2\bar{F}_3$ equals p_2. Define $F_3 : \mathcal{S} \to \mathcal{U}$ by $F_3 := \bar{F}_3\Pi$ and extend F_3 to a map on $\mathcal{X}$. Define $F := F_2+F_3$. Then $(A+BF)\mathcal{V} = (A + BF_2)\mathcal{V} \subset \mathcal{V}$ so $F \in \underline{F}(\mathcal{V})$. Also, $(A + BF) \mid \mathcal{S}/\mathcal{V} = A_2 + B_2\bar{F}_3$ since

$$(A_2 + B_2\bar{F}_3)\Pi = A_2\Pi + \Pi BF_3 = \Pi(A + BF_2) + \Pi BF_3 = \Pi(A + BF).$$

Thus, the characteristic polynomial of $(A + BF) \mid \mathcal{S}/\mathcal{V}$ is equal to p_2. It remains to be shown that the characteristic polynomial of $(A + BF) \mid \mathcal{R}$ equals p_1. This however follows from the fact that $(A + BF) \mid \mathcal{R} = A_0 + B_0F_1$.

(iii) Let $F_1, F_2 \in \underline{F}(\mathcal{V})$. According to lemma 4.19 we have $(F_1-F_2)\mathcal{V} \subset B^{-1}\mathcal{V}$. Hence $B(F_1 - F_2)\mathcal{V} \subset \mathcal{V}\cap \operatorname{im} B \subset \mathcal{R}$ (see lemma 4.16). Let $\Pi_1 : \mathcal{V} \to \mathcal{V}/\mathcal{R}$ be the canonical projection. Since $\mathcal{R} = \ker \Pi_1$ we have $\Pi_1 B(F_1 - F_2)\mathcal{V} = 0$. Denote $A_1 := (A+BF_1) \mid \mathcal{V}/\mathcal{R}$ and $A_2 := (A+BF_2) \mid \mathcal{V}/\mathcal{R}$. Let $\bar{x} \in \mathcal{V}/\mathcal{R}$, say $\bar{x} = \Pi_1 x$ with $x \in \mathcal{V}$. Then $(A_1 - A_2)\bar{x} = (A_1\Pi_1 - A_2\Pi_1)x = (\Pi_1(A + BF_1) - \Pi_1(A + BF_2))x = \Pi_1 B(F_1 - F_2)x = 0$. Thus $A_1 = A_2$ and hence the map $(A + BF) \mid \mathcal{V}/\mathcal{R}$ is independent of F for $F \in \underline{F}(\mathcal{V})$.

Finally, let $\Pi_2 : \mathcal{X} \to \mathcal{X}/\mathcal{S}$ be the canonical projection. Since $\operatorname{im} B \subset \mathcal{S}$ and $\mathcal{S} = \ker \Pi_2$ we have $\Pi_2 B = 0$. Let F_1, F_2 be linear maps $\mathcal{X} \to \mathcal{U}$. Note that $\mathcal{S}$ is $(A + BF_i)$-invariant $(i = 1, 2)$ and define $A_i := (A + BF_i) \mid \mathcal{X}/\mathcal{S}$. Let $\bar{x} \in \mathcal{X}/\mathcal{S}$, say $\bar{x} = \Pi_2 x$. Then $(A_1 - A_2)\bar{x} = (A_1\Pi_2 - A_2\Pi_2)x = (\Pi_2(A + BF_1) - \Pi_2(A + BF_2))x = \Pi_2 B(F_1 - F_2)x = 0$. Thus $A_1 = A_2$ and $(A + BF) \mid \mathcal{X}/\mathcal{S} = A \mid \mathcal{X}/\mathcal{S}$ for every F. ■

We close this section by applying theorem 4.18 to obtain the following characterization of controllability subspaces.

Theorem 4.20 *Consider the system (4.1). Let $\mathcal{V}$ be a subspace of $\mathcal{X}$. The following statements are equivalent:*

(*i*) *$\mathcal{V}$ is a controllability subspace,*

(*ii*) *for all for all $\lambda \in \mathbb{C}$ we have $(\lambda I - A)\mathcal{V} + \operatorname{im} B = \mathcal{V} + \operatorname{im} B$,*

(*iii*) *for each real monic polynomial p with $\deg p = \dim \mathcal{V}$, there exists $F \in \underline{F}(\mathcal{V})$ such that the characteristic polynomial of $A_F \mid \mathcal{V}$ equals p.*

Proof : (i) $\Rightarrow$ (iii). Follows by applying theorem 4.18 to the subspace $\mathcal{V}$. Note that $\mathcal{R}^*(\mathcal{V}) = \mathcal{V}$.

(iii) $\Rightarrow$ (ii). From the fact that $\underline{F}(\mathcal{V}) \neq \varnothing$ it follows that $\mathcal{V}$ is controlled invariant. Hence $A\mathcal{V} \subset \mathcal{V} + \operatorname{im} B$ (see theorem 4.2) and consequently $(\lambda I - A)\mathcal{V} + \operatorname{im} B \subset \mathcal{V} + \operatorname{im} B$ for all $\lambda \in \mathbb{C}$. To prove the converse inclusion it suffices to show that $\mathcal{V} \subset (\lambda I - A)\mathcal{V} + \operatorname{im} B$ for all λ. Let $\lambda \in \mathbb{C}$. Pick a real monic polynomial p with $\deg p = \dim \mathcal{V}$ such that $p(\lambda) \neq 0$. There is $F \in \underline{F}(\mathcal{V})$ such that the characteristic polynomial of $A_F \mid \mathcal{V}$ equals p. It follows that $\lambda \notin \sigma(A + BF \mid \mathcal{V})$ so the map $(\lambda I - A - BF) \mid \mathcal{V}$ must be regular. Also, $(\lambda I - A - BF)\mathcal{V} \subset \mathcal{V}$ and consequently we must in fact have $(\lambda I - A - BF)\mathcal{V} = \mathcal{V}$. It follows that $\mathcal{V} \subset (\lambda I - A)\mathcal{V} + \operatorname{im} B$.

(ii) $\Rightarrow$ (i). If (ii) holds then also $A\mathcal{V} \subset \mathcal{V} + \operatorname{im} B$ so $\mathcal{V}$ is controlled invariant. For each linear map F we have

$$(\lambda I - A - BF)\mathcal{V} + \operatorname{im} B = \mathcal{V} + \operatorname{im} B \tag{4.16}$$

for all $\lambda \in \mathbb{C}$. If in (4.16) we take $F \in \underline{F}(\mathcal{V})$ and intersect both sides of the equation with $\mathcal{V}$ we obtain

$$(\lambda I - A - BF)\mathcal{V} + (\operatorname{im} B \cap \mathcal{V}) = \mathcal{V} \tag{4.17}$$

for all $\lambda \in \mathbb{C}$. Let L be a linear map such that $\operatorname{im} L = B^{-1}\mathcal{V}$, say $L : \mathbb{R}^k \to \mathcal{U}$. Then $\operatorname{im} B \cap \mathcal{V} = \operatorname{im} BL$ and (4.17) becomes

$$(\lambda I - A - BF)\mathcal{V} + BL\mathbb{R}^k = \mathcal{V} \tag{4.18}$$

for all λ. By theorem 3.13 (compare (3.12)) this implies that the system $(A+BF, BL)$ with state space $\mathcal{V}$ and input space $\mathbb{R}^k$ is controllable. Hence, by corollary 3.4, $\mathcal{V} = \langle A + BF \mid \operatorname{im} BL\rangle$. Finally, apply theorem 4.12. ■

4.6 Stabilizability subspaces

In this section we introduce the notion of stabilizability subspace. Consider the system (4.1). From section 2.6, recall that if we choose an input function that is Bohl, then for any initial state also the resulting state trajectory is Bohl. Moreover, for a Bohl trajectory x we defined its spectrum $\sigma(x)$ and we called a Bohl trajectory x stable with respect to a given stability domain $\mathbb{C}_g$ if $\sigma(x) \subset \mathbb{C}_g$.

Let $\mathbb{C}_g$ be a stability domain. A subspace $\mathcal{V}$ of the state space is called a stabilizability subspace if it has the following property: for each initial condition in the subspace there is a Bohl input such that the resulting state trajectory remains in the subspace and is stable.

Definition 4.21 *A subspace $\mathcal{V} \subset \mathcal{X}$ is called a* stabilizability subspace *if for any $x_0 \in \mathcal{V}$ there exists a Bohl function u such that $x_u(t, x_0) \in \mathcal{V}$ for all $t \geqslant 0$ and $x_u(\cdot, x_0)$ is stable.*

An important special case is obtained by taking the stability set $\mathbb{C}_g$ to be equal to $\mathbb{C}^-$, the open left half complex plane. Since a Bohl function converges to zero as t tends to infinity if and only if its spectrum is contained in $\mathbb{C}^-$, for this particular case the requirement that $x_u(\cdot, x_0)$ should be stable is equivalent to the condition $x_u(t, x_0) \to 0$ $(t \to \infty)$.

Note that every stabilizability subspace is controlled invariant. The sum of any number of stabilizability subspaces is a stabilizability subspace. It also follows from the definition that the property of being a stabilizability subspace is invariant under state feedback and isomorphisms of the input space. In the following, for a given stability domain $\mathbb{C}_g$, let $\mathbb{C}_b$ be its complement in $\mathbb{C}$. Stabilizability subspaces can be characterized as follows:

Theorem 4.22 *Consider the system (4.1) and let $\mathbb{C}_g$ be a stability domain. Let $\mathcal{V}$ be a subspace of $\mathcal{X}$. Then the following statements are equivalent:*

(*i*) *$\mathcal{V}$ is a stabilizability subspace,*

(*ii*) *for all $\lambda \in \mathbb{C}_b$ we have $(\lambda I - A)\mathcal{V} + \operatorname{im} B = \mathcal{V} + \operatorname{im} B$,*

(*iii*) *there exists $F \in \underline{F}(\mathcal{V})$ such that $\sigma(A_F \mid \mathcal{V}) \subset \mathbb{C}_g$.*

Proof : (i) $\Rightarrow$ (ii). For any F, condition (ii) is equivalent to

$$(\lambda I - A_F)\mathcal{V} + \operatorname{im} B = \mathcal{V} + \operatorname{im} B \quad \text{for all } \lambda \in \mathbb{C}_b. \tag{4.19}$$

Let $F \in \underline{F}(\mathcal{V})$. We claim that in this case (4.19) is equivalent to

$$(\lambda I - A_F)\mathcal{V} + (\operatorname{im} B \cap \mathcal{V}) = \mathcal{V} \quad \text{for all } \lambda \in \mathbb{C}_b. \tag{4.20}$$

Indeed, (4.20) follows from (4.19) by taking the intersection with $\mathcal{V}$ on both sides of the equation. The converse can be verified immediately. Now assume that (ii) does not hold. Then by the previous there must be a $\lambda_0 \in \mathbb{C}_b$ for which the equality in (4.20) does not hold. Since $F \in \underline{F}(\mathcal{V})$ we do have

$$(\lambda_0 I - A_F)\mathcal{V} + (\operatorname{im} B \cap \mathcal{V}) \subset \mathcal{V}. \tag{4.21}$$

Consequently, the inclusion in (4.21) must be *strict*. Hence, there exists a nonzero row vector η such that $\eta \perp (\lambda_0 I - A_F)\mathcal{V} + (\operatorname{im} B \cap \mathcal{V})$ but *not* $\eta \perp \mathcal{V}$. Let $x_0 \in \mathcal{V}$ such that $\eta x_0 \neq 0$. Since $\mathcal{V}$ is a stabilizability subspace there exists a Bohl function u such that if $x(t)$ satisfies $\dot{x} = A_F x + Bu, x(0) = x_0$, we have $x(t) \in \mathcal{V}$ for all $t \geqslant 0$ and $\sigma(x) \subset \mathbb{C}_g$. Since $x(t) \in \mathcal{V}$ for all $t \geqslant 0$, also $\dot{x}(t) \in \mathcal{V}$ and $A_F x(t) \in \mathcal{V}$ for all $t \geqslant 0$. Hence, $Bu(t) \in \mathcal{V} \cap \operatorname{im} B$ for all $t \geqslant 0$ and therefore $\eta Bu(t) = 0,\ t \geqslant 0$. Also, since $\eta \perp (\lambda_0 I - A_F)\mathcal{V}$, we have $\eta(\lambda_0 I - A_F)x(t) = 0$ for all t and hence $\eta A_F x(t) = \lambda_0 \eta x(t)$ for all t. Now define $z(t) := \eta x(t)$. Then $z(0) = \eta x_0 \neq 0$ and z satisfies the differential equation $\dot{z}(t) = \lambda_0 z(t)$. It follows that $\eta^* x(t) = z(t) = e^{\lambda_0 t}\eta^* x_0$. Thus $\lambda_0 = \sigma(\eta^* x)$. Obviously, $\sigma(\eta^* x) \subset \sigma(x) \subset \mathbb{C}_g$. Since $\lambda_0 \in \mathbb{C}_b$, this yields a contradiction.

(ii) $\Rightarrow$ (iii). If (ii) holds then $A\mathcal{V} \subset \mathcal{V} + \operatorname{im} B$ so $\mathcal{V}$ is controlled invariant. Take $F_0 \in \underline{F}(\mathcal{V})$. Then (ii) is equivalent to (4.20) with F replaced by F_0. Let L be a linear map such that $\operatorname{im} L = B^{-1}\mathcal{V}$, say $L : \mathbb{R}^k \to \mathcal{U}$. Then $\operatorname{im} B \cap \mathcal{V} = \operatorname{im} BL$ so (4.20) yields $(\lambda I - A_{F_0})\mathcal{V} + BL\mathbb{R}^k = \mathcal{V}$ for all $\lambda \in \mathbb{C}_b$. It follows from theorem 3.13 that the system (A_{F_0}, BL) with state space $\mathcal{V}$ and input space $\mathbb{R}^k$ is stabilizable. Hence there is $F_1 : \mathcal{V} \to \mathbb{R}^k$ such that $\sigma((A_{F_0} + BLF_1) \mid \mathcal{V}) \subset \mathbb{C}_g$. Extend F_1 to a linear map on $\mathcal{X}$ and define $F := F_0 + LF_1$. Then $\sigma(A_F \mid \mathcal{V}) \subset \mathbb{C}_g$ and since $\operatorname{im}(F - F_0) \subset B^{-1}\mathcal{V}$ also $F \in \underline{F}(\mathcal{V})$ (see lemma 4.19).

(iii) $\Rightarrow$ (i). Let $F \in \underline{F}(\mathcal{V})$ with $\sigma(A_F \mid \mathcal{V}) \subset \mathbb{C}_g$. Denote $A_0 := A_F \mid \mathcal{V}$ and apply state feedback $u = Fx$. The resulting state trajectory is given by $x(t) = e^{A_0 t}x_0$. Obviously for $x_0 \in \mathcal{V}$ we have that $x(t) \in \mathcal{V}$ for all t. Then it follows from theorem 2.6 that the spectrum of x must be contained in $\sigma(A_0)$ which is contained in $\mathbb{C}_g$. Finally note that x is equal to the state trajectory resulting from the Bohl input $u(t) = Fe^{A_0 t}x_0$. ■

As already noted in the proof of the previous theorem, for any $F \in \underline{F}(\mathcal{V})$ and any map L such that $\operatorname{im} L = B^{-1}\mathcal{V}$ condition (ii) is equivalent to saying that the system $\dot{x} = A_F x + BLw$ with state space $\mathcal{V}$ is stabilizable. In this sense, a stabilizability subspace can be considered as a subspace to which the original system can be restricted by suitable restriction of the input functions, such that the restricted system is stabilizable. From theorem 4.20 it follows that, in the same sense, a controllability subspace is a subspace for which the restricted system is controllable. Note that, given any stability domain $\mathbb{C}_g$, every controllability subspace is a stabilizability subspace.

Using the feedback characterization in theorem 4.22 (iii) it is possible to characterize stabilizability subspaces in terms of the spectral assignability properties of controlled invariant subspaces studied in the previous section. For a given controlled invariant subspace $\mathcal{V}$, denote $\mathcal{R} := \mathcal{R}^*(\mathcal{V})$. It was shown in theorem 4.18 that the map $A_F \mid \mathcal{V}/\mathcal{R}$ is independent of F for $F \in \underline{F}(\mathcal{V})$. If $\mathcal{V}$ is a stabilizability subspace then there exists $F \in \underline{F}(\mathcal{V})$ such that $\sigma(A_F \mid \mathcal{V}) \subset \mathbb{C}_g$. Since

$$\sigma(A_F \mid \mathcal{V}) = \sigma(A_F \mid \mathcal{R}) \cup \sigma(A_F \mid \mathcal{V}/\mathcal{R})$$

this implies that the fixed spectrum $\sigma(A_F \mid \mathcal{V}/\mathcal{R})$ is contained in $\mathbb{C}_g$. Conversely, if $\mathcal{V}$ is a controlled invariant subspace such that the fixed spectrum $\sigma(A_F \mid \mathcal{V}/\mathcal{R}) \subset \mathbb{C}_g$ then obviously one can find a $F_1 \in \underline{F}(\mathcal{V})$ such that $\sigma(A_{F_1} \mid \mathcal{V}) \subset \mathbb{C}_g$ (since the characteristic polynomial of $A_{F_1} \mid \mathcal{R}$ can be chosen arbitrarily). Thus we have shown:

Corollary 4.23 *Let $\mathcal{V}$ be a controlled invariant subspace. Then $\mathcal{V}$ is a stabilizability subspace if and only if $\sigma(A + BF \mid \mathcal{V}/\mathcal{R}) \subset \mathbb{C}_g$ for any $F \in \underline{F}(\mathcal{V})$.*

If $\mathcal{K}$ is an arbitrary subspace then we want to consider the largest stabilizability subspace contained in $\mathcal{K}$.

Definition 4.24 *Let $\mathbb{C}_g$ be a stability set and let $\mathcal{K}$ be a subspace of $\mathcal{X}$. Then we define*

$$\mathcal{V}_g^*(\mathcal{K}) := \{x_0 \in \mathcal{X} \mid \text{ there is a Bohl function } u \text{ such that } x_u(t, x_0) \in \mathcal{K} \text{ for all } t \geqslant 0 \text{ and } x_u(\cdot, x_0) \text{ is } \mathbb{C}_g\text{-stable.}\}$$

Theorem 4.25 *Let $\mathcal{K}$ be a subspace of $\mathcal{X}$. Then $\mathcal{V}_g^*(\mathcal{K})$ is the largest stabilizability subspace contained in $\mathcal{K}$, i.e.,*

(*i*) *$\mathcal{V}_g^*(\mathcal{K})$ is a stabilizability subspace,*

(*ii*) *$\mathcal{V}_g^*(\mathcal{K}) \subset \mathcal{K}$,*

(*iii*) *if $\mathcal{V} \subset \mathcal{K}$ is a stabilizability subspace then $\mathcal{V} \subset \mathcal{V}_g^*(\mathcal{K})$.*

The proof is similar to that of theorem 4.5 and is left as an exercise to the reader.

Sometimes we denote $\mathcal{V}_g^*(\mathcal{K})$ by $\mathcal{V}_g^*(\mathcal{K}, A, B)$. It is easily verified that for a given subspace $\mathcal{K}$ the following relation holds:

$$\mathcal{R}^*(\mathcal{K}) \subset \mathcal{V}_g^*(\mathcal{K}) \subset \mathcal{V}^*(\mathcal{K}) \subset \mathcal{K}.$$

More specifically, we have $\mathcal{R}^*(\mathcal{V}_g^*(\mathcal{K})) = \mathcal{R}^*(\mathcal{K})$ and $\mathcal{V}_g^*(\mathcal{V}^*(\mathcal{K})) = \mathcal{V}_g^*(\mathcal{K})$. In particular, if we take $\mathcal{K} = \mathcal{X}$ we obtain the largest stabilizability subspace of the system (4.1), $\mathcal{V}_g^*(\mathcal{X})$. This subspace will be called the *stabilizable subspace of* (A, B) and will be denoted by $\mathcal{X}_{\text{stab}}$ or $\mathcal{X}_{\text{stab}}(A, B)$ (see also exercise 3.24). This subspace consists exactly of those points in which a stable state trajectory starts:

$$\mathcal{X}_{\text{stab}} = \{x_0 \in \mathcal{X} \mid \text{there is a Bohl function } u \text{ such that } x_u(\cdot, x_0) \text{ is stable.}\} \tag{4.22}$$

According to the following result, the stabilizable subspace is equal to the sum of the stable subspace of A (see definition 2.13) and the reachable subspace of (A, B):

Theorem 4.26 $\mathcal{X}_{\text{stab}} = \mathcal{X}_g(A) + \langle A \mid \text{im}\, B\rangle$.

Proof : ($\supset$). Obviously, both $\mathcal{X}_g(A)$ and $\langle A \mid \text{im}\, B\rangle$ are stabilizability subspaces and hence also their sum. This sum is contained in the largest stabilizability subspace.

($\subset$) In this proof, denote $\mathcal{V} := \mathcal{X}_g(A) + \langle A \mid \text{im}\, B\rangle$. It is easily seen that both $\mathcal{X}_{\text{stab}}$ as well as $\mathcal{V}$ are A-invariant. Let $\Pi : \mathcal{X}_{\text{stab}} \to \mathcal{X}_{\text{stab}}/\mathcal{V}$ be the canonical projection and denote $A_0 := A \mid \mathcal{X}_{\text{stab}}/\mathcal{V}$. We claim that $\sigma(A_0) \subset \mathbb{C}_b$. Indeed,

$$\sigma(A \mid \mathcal{X}_{\text{stab}}/\mathcal{V}) \subset \sigma(A \mid \mathcal{X}/\mathcal{V}) \subset \sigma(A \mid \mathcal{X}/\mathcal{X}_g(A)).$$

The latter spectrum is of course equal to $\sigma(A \mid \mathcal{X}_b(A))$, which is contained in $\mathbb{C}_b$. Now assume that $\mathcal{V} \subset \mathcal{X}_{\text{stab}}$ with *strict* inclusion. Then there is $x_0 \in \mathcal{X}_{\text{stab}}$ with $x_0 \notin \mathcal{V}$. By (4.22) there is a Bohl function u such that the resulting trajectory x is stable. Let $\bar{x}(t) := \Pi x(t)$. Then $\bar{x}$ satisfies

$$\dot{\bar{x}} = \Pi \dot{x} = \Pi A x + \Pi B u = A_0 \Pi x = A_0 \bar{x}.$$

Here we used the facts that $\operatorname{im} B \subset \mathcal{V} = \ker \Pi$ and $\Pi A = A_0 \Pi$. Since $\bar{x}(0) = \Pi x_0 \neq 0$, $\bar{x}$ is unstable. This contradicts the assumption that x and hence Πx is stable. ■

The above result can also be used to obtain an expression for the largest stabilizability subspace contained in an arbitrary subspace of the state space. Let $\mathcal{K}$ be a subspace of $\mathcal{X}$ and let $\mathcal{V}^*$ be the largest controlled invariant subspace in $\mathcal{K}$. Take an arbitrary $F \in \underline{F}(\mathcal{V}^*)$ and let L be a linear map such that $\operatorname{im} L = B^{-1}\mathcal{V}^*$, say $L : \mathbb{R}^k \to \mathcal{U}$. Consider the restricted system $\dot{x}(t) = A_F x(t) + BLw(t)$ with state space $\mathcal{V}^*$ and input space $\mathbb{R}^k$. Temporarily, denote the stabilizable subspace of the restricted system by $\tilde{\mathcal{X}}_{\text{stab}}$. By (4.22),

$$\tilde{\mathcal{X}}_{\text{stab}} = \{x_0 \in \mathcal{V}^* \mid \text{ there is a Bohl function } w \text{ such that the solution } x(t) \text{ of } \dot{x} = A_F x + BLw,\ x(0) = x_0 \text{ is stable}\}.$$

We claim that the largest stabilizability subspace in $\mathcal{K}$ is equal to the stabilizable subspace of the restricted system, i.e. $\mathcal{V}_g^*(\mathcal{K}) = \tilde{\mathcal{X}}_{\text{stab}}$. To prove this, first recall that $\mathcal{V}_g^*(\mathcal{K}) = \mathcal{V}_g^*(\mathcal{V}^*)$. Let $x_0 \in \mathcal{V}_g^*(\mathcal{K})$. By definition 4.24 there exists a Bohl function u such that $x_u(t, x_0) \in \mathcal{V}^*$ for all $t \geqslant 0$ and $x_u(\cdot, x_0)$ is $\mathbb{C}_g$-stable. It follows from theorem 4.3 that the control u must be of the form

$$u(t) = F x_u(t, x_0) + Lw(t)$$

for some Bohl function w. Thus $x_u(t, x_0)$ satisfies $\dot{x} = A_F x + BLw, x(0) = x_0$. This shows that $x_0 \in \tilde{\mathcal{X}}_{\text{stab}}$. The converse inclusion, i.e. the inclusion $\tilde{\mathcal{X}}_{\text{stab}} \subset \mathcal{V}_g^*(\mathcal{K})$ is left as an exercise to the reader. By applying theorem 4.26 we find that

$$\mathcal{V}_g^*(\mathcal{K}) = \mathcal{X}_g(A_F \mid \mathcal{V}^*) + \langle A_F \mid \operatorname{im} BL \rangle.$$

In theorem 4.17 it was shown that $\langle A_F \mid \operatorname{im} BL \rangle$ (the reachable subspace of the restricted system) is equal to $\mathcal{R}^*(\mathcal{K})$. Moreover, since $\mathcal{V}^*$ is invariant under $A + BF$, we have

$$\mathcal{X}_g(A_F \mid \mathcal{V}^*) = \mathcal{X}_g(A_F) \cap \mathcal{V}^*.$$

Thus we obtain the following characterization of the largest stabilizability subspace contained in a given subspace:

Corollary 4.27 *Let $\mathcal{K}$ be a subspace of $\mathcal{X}$. Then for all $F \in \underline{F}(\mathcal{V}^*(\mathcal{K}))$ we have*

$$\mathcal{V}_g^*(\mathcal{K}) = \mathcal{X}_g(A + BF) \cap \mathcal{V}^*(\mathcal{K}) + \mathcal{R}^*(\mathcal{K}).$$

In theorem 4.22 we characterized stabilizability subspaces as controlled invariant subspaces $\mathcal{V}$ for which there exist $F \in \underline{F}(\mathcal{V})$ such that the restriction of $A + BF$ to $\mathcal{V}$ has all its eigenvalues in $\mathbb{C}_g$. Sometimes it will be important to know something about the spectrum of the map induced by $A + BF$ on the factor space $\mathcal{X}/\mathcal{V}$. A controlled invariant subspace $\mathcal{V}$ will be called *outer stabilizable* if there exists an $F \in \underline{F}(\mathcal{V})$ such that $\sigma(A + BF \mid \mathcal{X}/\mathcal{V}) \subset \mathbb{C}_g$. Using the terminology of section 2.7 this can be stated alternatively as: if there exists $F \in \underline{F}(\mathcal{V})$ such that the subspace $\mathcal{V}$ is outer stable with respect to the map $A + BF$. Correspondingly, stabilizability subspaces will sometimes be called *inner stabilizable* controlled invariant subspaces. In order to illustrate these concepts, let $\mathcal{V}$ be a controlled invariant subspace for the system (4.1). Choose a basis for $\mathcal{X}$ adapted to $\mathcal{V}$. Accordingly, split

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Now, first assume that $\mathcal{V}$ is inner stabilizable. Then there is a map $F = (F_1 \;\; F_2)$ such that $A_{21} + B_2 F_1 = 0$ and $\sigma(A_{11} + B_1 F_1) \subset \mathbb{C}_g$. By taking $F_2 = 0$ we thus obtain

$$A + BF = \begin{pmatrix} A_{11} + B_1 F_1 & A_{12} \\ 0 & A_{22} \end{pmatrix}.$$

Let $x_0 \in \mathcal{V}$. Then with respect to the above choice of basis for $\mathcal{X}$ we have $x_0 = (x_{10}^{\mathrm{T}}, 0)^{\mathrm{T}}$. Apply the state feedback control $u = Fx$. The trajectory resulting from x_0 and u satisfies the equations

$$\begin{aligned} \dot{x}_1(t) &= (A_{11} + B_1 F_1)x_1(t) + A_{12}x_2(t), \;\; x_1(0) = x_{10}, \\ \dot{x}_2(t) &= A_{22}x_2(t), \;\; x_2(0) = 0. \end{aligned}$$

Thus $x_2(t) = 0$ $(t \geqslant 0)$ and $x_1(t) = e^{(A_{11}+B_1F_1)t}x_{10}$ $(t \geqslant 0)$. The fact that $x_2(t) = 0$ for $t \geqslant 0$ expresses the fact that the trajectory $x_u(t, x_0)$ remains in $\mathcal{V}$ for all $t \geqslant 0$. The expression for $x_1(t)$ displays the fact that the spectrum of $x_u(\cdot, x_0)$ lies in $\mathbb{C}_g$. In particular, if $\mathbb{C}_g = \mathbb{C}^-$ then $x_1(t) \to 0$ as $t \to \infty$.

Next, instead of inner stabilizable, let us assume that $\mathcal{V}$ is outer stabilizable. Then there is a map $F = (F_1 \;\; F_2)$ such that $A_{21} + B_2 F_1 = 0$ and $\sigma(A_{22} + B_2 F_2) \subset \mathbb{C}_g$. Thus

$$A + BF = \begin{pmatrix} A_{11} + B_1 F_1 & A_{12} + B_1 F_2 \\ 0 & A_{22} + B_2 F_2 \end{pmatrix}.$$

Take any $x_0 = (x_{10}^{\mathrm{T}}, x_{20}^{\mathrm{T}})^{\mathrm{T}} \in \mathcal{X}$. Apply the state feedback control $u = Fx$. Then the trajectory resulting from x_0 and u is given by

$$\begin{aligned} \dot{x}_1(t) &= (A_{11} + B_1 F_1)x_1(t) + (A_{12} + B_1 F_2)x_2(t), \;\; x_1(0) = x_{10}, \\ \dot{x}_2(t) &= (A_{22} + B_2 F_2)x_2(t), \;\; x_2(0) = x_{20}. \end{aligned}$$

Consequently, $x_2(t) = e^{(A_{22}+B_2F_2)t}x_{20}$. Thus, if we assume that $\mathbb{C}_g = \mathbb{C}^-$ then $x_2(t) \to 0$ $(t \to \infty)$. This expresses the fact that the trajectory $x_u(t, x_0)$ converges to the subspace $\mathcal{V}$ as $t \to \infty$. We see that, assuming that $\mathbb{C}_g = \mathbb{C}^-$, an outer stabilizable controlled invariant subspace has the property that there exists a state feedback such that all trajectories starting in the subspace remain in it, while all other trajectories converge to that subspace as $t \to \infty$. An inner stabilizable subspace has the property that there exists a state feedback such that all trajectories starting in the subspace remain in the subspace and converge to the origin as $t \to \infty$.

We will now establish a criterion for a controlled invariant subspace to be outer stabilizable. In the following, again let $\mathbb{C}_g$ be an arbitrary stability domain.

Lemma 4.28 *Let $\mathcal{V}$ be a controlled invariant subspace. Then $\mathcal{V}$ is outer stabilizable if and only if*

$$\sigma\,(A \mid \mathcal{X}/(\mathcal{V} + \langle A \mid \operatorname{im} B\rangle)) \subset \mathbb{C}_g.$$

Proof : Denote $\mathcal{S} := \mathcal{V} + \langle A \mid \operatorname{im} B\rangle$.

($\Rightarrow$) Let $F \in \underline{F}(\mathcal{V})$ be a map such that $\sigma(A_F \mid \mathcal{X}/\mathcal{V}) \subset \mathbb{C}_g$. Since both $\mathcal{S}$ and $\mathcal{V}$ are invariant under A_F and since, by theorem 4.18, $A_F \mid \mathcal{X}/\mathcal{S}$ is equal to $A \mid \mathcal{X}/\mathcal{S}$ we have

$$\sigma(A \mid \mathcal{X}/\mathcal{S}) = \sigma(A_F \mid \mathcal{X}/\mathcal{S}) \subset \sigma(A_F \mid \mathcal{X}/\mathcal{V}) \subset \mathbb{C}_g.$$

($\Leftarrow$) Let p be a real monic polynomial with all its zeros in $\mathbb{C}_g$, with $\deg p = \dim \mathcal{S}/\mathcal{V}$. According to theorem 4.18 there is $F \in \underline{F}(\mathcal{V})$ such that the characteristic polynomial of $A_F \mid \mathcal{S}/\mathcal{V}$ equals p. Hence $\sigma(A_F \mid \mathcal{S}/\mathcal{V}) \subset \mathbb{C}_g$. It follows that

$$\begin{aligned}\sigma(A_F \mid \mathcal{X}/\mathcal{V}) &= \sigma(A_F \mid \mathcal{X}/\mathcal{S}) \cup \sigma(A_F \mid \mathcal{S}/\mathcal{V})\\ &= \sigma(A \mid \mathcal{X}/\mathcal{S}) \cup \sigma(A_F \mid \mathcal{S}/\mathcal{V}) \subset \mathbb{C}_g.\end{aligned}$$

■

Theorem 4.29 *Let $\mathcal{V}$ be a controlled invariant subspace. Then $\mathcal{V}$ is outer stabilizable if and only if*

$$\mathcal{V} + \mathcal{X}_{\text{stab}} = \mathcal{X}. \tag{4.23}$$

Proof : ($\Rightarrow$) Assume that (4.23) does not hold. By theorem 4.26 we then have $\mathcal{S} + \mathcal{X}_g(A) \subset \mathcal{X}$ with *strict* inclusion. Thus

$$\sigma\left(A \mid \mathcal{X}/(\mathcal{S} + \mathcal{X}_g(A))\right) \neq \varnothing. \tag{4.24}$$

The spectrum (4.24) is contained in $\sigma(A \mid \mathcal{X}/\mathcal{S})$. By lemma 4.28, the latter is contained in $\mathbb{C}_g$. On the other hand, the spectrum (4.24) is contained in $\sigma(A \mid \mathcal{X}/\mathcal{X}_g(A))$, which is contained in $\mathbb{C}_b$. This yields a contradiction.

($\Leftarrow$) Assume that (4.23) holds. Then we have

$$\begin{aligned}\sigma(A \mid \mathcal{X}/\mathcal{S}) &= \sigma(A \mid (\mathcal{S} + \mathcal{X}_g(A))/\mathcal{S}) \\ &= \sigma(A \mid \mathcal{X}_g(A)/(\mathcal{S} \cap \mathcal{X}_g(A))) \\ &\subset \sigma(A \mid \mathcal{X}_g(A)) \subset \mathbb{C}_g.\end{aligned}$$

It follows from (4.28) that $\mathcal{V}$ is outer stabilizable. ■

To conclude this section we will study the connection between the concepts introduced here and stabilizability of the system (4.1). Recall from section 3.10 that (A, B) is called stabilizable if there exists an F such that $A + BF$ is stable. Obviously, (A, B) is stabilizable if and only if the state space $\mathcal{X}$ is inner stabilizable. According to theorem 4.22 this is equivalent with $(\lambda I - A)\mathcal{X} + B\mathcal{U} = \mathcal{X}$ for all $\lambda \in \mathbb{C}_b$. Thus we recover theorem 3.32. On the other hand, (A, B) is stabilizable if and only if the zero-subspace is outer stabilizable. Using this observation we obtain

Theorem 4.30 *The following statements are equivalent:*

(*i*) (A, B) *is stabilizable,*

(*ii*) $\sigma(A \mid \mathcal{X}/\langle A \mid \operatorname{im} B\rangle) \subset \mathbb{C}_g$,

(*iii*) $\mathcal{X}_g(A) + \langle A \mid \operatorname{im} B\rangle = \mathcal{X}$,

(*iv*) $\mathcal{X}_b(A) \subset \langle A \mid \operatorname{im} B\rangle$.

Proof : The equivalence of (i), (ii) and (iii) follows immediately from theorem 4.26 and lemma 4.28. The equivalence of (iii) and (iv) follows from exercise 2.11. ■

4.7 Disturbance decoupling with internal stability

In this section we again consider the disturbed control system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + Ed(t), \\ z(t) &= Hx(t).\end{aligned} \tag{4.25}$$

In section 4.2 we discussed the problem of disturbance decoupling by state feedback. The problem was to find a state feedback control law $u(t) = Fx(t)$ such that the impulse response matrix

$$T_F(t) := He^{(A+BF)t}E$$

of the closed-loop system (4.6) is identically equal to zero (or, equivalently, such that the closed-loop transfer function: $G_F(s)$ is equal to zero). On the other hand, in section 3.10 we discussed stabilizability of the system (4.1), that is, the existence of a state feedback such that the system controlled by means of this feedback control law becomes internally stable. In the present section we combine these two requirements into one single design problem, the problem of disturbance decoupling with internal stability by state feedback. This problem will consist of finding a state feedback control law such that the closed-loop system (4.6) is disturbance decoupled and internally stable:

Definition 4.31 *Consider the system (4.25). Let* $\mathbb{C}_g$ *be a stability domain.* The problem of disturbance decoupling with internal stability by state feedback*, DDPS, is to find a linear map* $F : \mathcal{X} \to \mathcal{U}$ *such that* $T_F = 0$ *and* $\sigma(A + BF) \subset \mathbb{C}_g$.

Given a feedback map F, consider the closed-loop system (4.6). Using (4.6) we see that the closed-loop system is disturbance decoupled and internally stable if and only if there exists an A_F-invariant subspace $\mathcal{V}$ between $\operatorname{im} E$ and $\ker H$ and A_F is stable. If a subspace $\mathcal{V}$ is A_F-invariant and if A_F is stable then of course $\mathcal{V}$ is a stabilizability subspace and (A, B) is stabilizable. The following result states that the converse also holds:

Theorem 4.32 *Let* $\mathcal{V}$ *be a subspace of* $\mathcal{X}$*. There exists an* $F \in \underline{F}(\mathcal{V})$ *such that* $\sigma(A+BF) \subset \mathbb{C}_g$ *if and only if* $\mathcal{V}$ *is a stabilizability subspace and* (A, B) *is stabilizable.*

Proof : ($\Rightarrow$) Of course, (A, B) is stabilizable. Also $\sigma(A_F \mid \mathcal{V}) \subset \sigma(A_F) \subset \mathbb{C}_g$ so $\mathcal{V}$ is a stabilizability subspace.

($\Leftarrow$) Denote $\mathcal{S} := \mathcal{V} + \langle A \mid \operatorname{im} B\rangle$ and $\mathcal{R} := \mathcal{R}^*(\mathcal{V})$. It follows from theorem 4.18 that there exists an $F \in \underline{F}(\mathcal{V})$ such that $\sigma(A_F \mid \mathcal{R}) \subset \mathbb{C}_g$ and $\sigma(A_F \mid \mathcal{S}/\mathcal{V}) \subset \mathbb{C}_g$. Since $\mathcal{V}$ is a stabilizability subspace, according to corollary 4.23 we have $\sigma(A_F \mid \mathcal{V}/\mathcal{R}) \subset \mathbb{C}_g$. Finally, since (A, B) is stabilizable, using theorem 4.30 we obtain

$$\sigma(A \mid \mathcal{X}/\mathcal{S}) \subset \sigma(A \mid \mathcal{X}/\langle A \mid \operatorname{im} B\rangle) \subset \mathbb{C}_g. \qquad \blacksquare$$

Corollary 4.33 *There exists a linear map* $F : \mathcal{X} \to \mathcal{U}$ *such that* $T_F = 0$ *and* $\sigma(A + BF) \subset \mathbb{C}_g$ *if and only if there exists a stabilizability subspace* $\mathcal{V}$ *such that* $\operatorname{im} E \subset \mathcal{V} \subset \ker H$ *and* (A, B) *is stabilizable.*

Proof : ($\Rightarrow$) If $T_F = 0$ then (4.6) is disturbance decoupled. Hence there is an A_F-invariant subspace $\mathcal{V}$ with $\operatorname{im} E \subset \mathcal{V} \subset \ker H$. Since $\sigma(A_F) \subset \mathbb{C}_g$, $\mathcal{V}$ is a stabilizability subspace and (A, B) is stabilizable.

($\Leftarrow$) According to theorem 4.32 there is an F such that $\mathcal{V}$ is A_F-invariant and $\sigma(A_F) \subset \mathbb{C}_g$. Since $\operatorname{im} E \subset \mathcal{V} \subset \ker H$, the system (4.6) is disturbance decoupled. It follows that $T_F = 0$. $\blacksquare$

Of course, if im E is contained in a stabilizability subspace that is contained in ker H, then it is also contained in the largest stabilizability subspace contained in ker H (see theorem 4.25). Thus we obtain

Corollary 4.34 *There exists a linear map* $F : \mathcal{X} \to \mathcal{U}$ *such that* $T_F = 0$ *and* $\sigma(A + BF) \subset \mathbb{C}_g$ *if and only if* $\operatorname{im} E \subset \mathcal{V}_g^*(\ker H)$ *and* (A, B) *is stabilizable.*

We conclude this section by noting that it is, in principle, possible to verify the subspace inclusion $\operatorname{im} E \subset \mathcal{V}_g^*(\ker H)$ computationally. Indeed, recall from corollary 4.27 that for any $F \in \underline{F}(\mathcal{V}^*(\ker H))$ we have

$$\mathcal{V}_g^*(\ker H) = \mathcal{X}_g(A_F) \cap \mathcal{V}^*(\ker H) + \mathcal{R}^*(\ker H).$$

Thus, given the system (4.25) and a stability domain $\mathbb{C}_g$, one could first calculate $\mathcal{V}^*(\ker H)$ using the algorithm described in section 4.3. Next, one could calculate an $F \in \underline{F}(\mathcal{V}^*(\ker H))$ and compute the subspace $\mathcal{X}_g(A_F)$. Finally, the subspace $\mathcal{R}^*(\ker H)$ could be computed using theorem 4.17. Of course, the above only provides a very rough, conceptual, algorithm. If one would actually want to verify the conditions of corollary 4.34 computationally, several questions concerning numerical stability would have to be taken into account.

4.8 External stabilization

Again consider the system (4.25). In section 4.2 it was shown that the condition

$$\operatorname{im} E \subset \mathcal{V}^*(\ker H) \tag{4.26}$$

is necessary and sufficient for the existence of a state feedback control law $u(t) = Fx(t)$ such that the transfer function of the closed-loop system becomes equal to zero. The output of the system then becomes independent of the disturbance input and, in particular, if the initial condition of the closed loop system is zero then the output will be equal to zero for all disturbances. Suppose now that condition (4.26) does not hold, so that disturbance decoupling by state feedback is not possible. In this section we will set ourselves a more modest objective and ask ourselves the question: when can we find a state feedback control law $u(t) = Fx(t)$ such that the closed-loop transfer function becomes stable. Equivalently: when can we make the closed-loop system (4.6) input/output stable by choosing F appropriately? The rationale behind this objective is of course that if the closed-loop system is stable then at least stable disturbances will result in stable outputs. If for example we take the stability set to be equal to $\mathbb{C}^-$ and if the initial condition of the closed-loop system is equal to zero, then $d(t) \to 0$ $(t \to \infty)$ will imply $z(t) \to 0$ $(t \to \infty)$ (see corollary 3.22). Also, bounded disturbances will at least result in bounded outputs.

Let us first consider the uncontrolled system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Ed(t), \\ z(t) &= Hx(t). \end{aligned} \tag{4.27}$$

Let $G(s) := H(Is - A)^{-1}E$ be the transfer function from d to z. The following lemma provides a useful sufficient condition for $G(s)$ to be stable:

Lemma 4.35 *Let $\mathbb{C}_g$ be a stability domain. Assume that there exist A-invariant subspaces $\mathcal{V}_1 \subset \mathcal{V}_2$ of $\mathcal{X}$ such that $\mathcal{V}_1 \subset \ker H$, $\operatorname{im} E \subset \mathcal{V}_2$ and $\sigma(A \mid \mathcal{V}_2/\mathcal{V}_1) \subset \mathbb{C}_g$. Then $G(s)$ is stable.*

Proof : Let $\Pi : \mathcal{V}_2 \to \mathcal{V}_2/\mathcal{V}_1$ be the canonical projection. Denote $\bar{A} := A \mid \mathcal{V}_2/\mathcal{V}_1$. Let $\bar{H} : \mathcal{V}_2/\mathcal{V}_1 \to \mathcal{Z}$ be a linear map such that $\bar{H}\Pi = H \mid \mathcal{V}_2$ (such a map $\bar{H}$ exists since $\mathcal{V}_1 \subset \ker H$, see section 2.4). Let $\bar{E} := \Pi E$. Then we have

$$G(s) = H(Is - A)^{-1}E = \bar{H}(Is - \bar{A})^{-1}\bar{E}.$$

(see exercise 3.8). Since $\sigma(\bar{A}) \subset \mathbb{C}_g$, we conclude that $G(s)$ is stable. ∎

In the following, let $G_F(s)$ be the transfer function of the closed loop system (4.6).

Definition 4.36 *Consider the system (4.3). Let $\mathbb{C}_g$ be a stability domain.* The problem of external stabilization by state feedback, *ESP, is to find a linear map $F : \mathcal{X} \to \mathcal{U}$ such that $G_F(s)$ is stable.*

Assume that F is a map such that $G_F(s)$ is stable. Then for every point $x_0 \in \operatorname{im} E$, $H(Is - A_F)^{-1}x_0$ is stable. This says that if in the system $\dot{x}(t) = Ax(t) + Bu(t)$ with initial condition $x(0) = x_0$ we use the control law $u(t) = Fx(t)$ then the resulting state trajectory $x_u(\cdot, x_0)$ has the property that $Hx_u(\cdot, x_0)$ is stable. Of course, $x_u(\cdot, x_0)$ also results from the open loop control $u(t) = Fe^{A_F t}x_0$. Thus we find that if there exists an F such that $G_F(s)$ is stable then $\operatorname{im} E$ must be contained in

$$\mathcal{W}_g(\ker H) := \{x_0 \in \mathcal{X} \mid \text{there is a Bohl function } u \text{ such that } Hx_u(\cdot, x_0) \text{ is stable}\}. \tag{4.28}$$

It is easy to verify that $\mathcal{W}_g(\ker H)$ is a subspace of $\mathcal{X}$. This subspace will turn out to play a central role in the problem of external stabilization. Often, we will denote $\mathcal{W}_g(\ker H)$ by $\mathcal{W}_g$. We have the following characterization of $\mathcal{W}_g(\ker H)$ in terms of controlled invariant subspaces introduced before:

Theorem 4.37 $\mathcal{W}_g(\ker H) = \mathcal{V}^*(\ker H) + \mathcal{X}_{\text{stab}}$.

Proof : ($\supset$) It follows immediately from definition 4.4 and (4.22) that both $\mathcal{V}^*(\ker H)$ and $\mathcal{X}_{\text{stab}}$ are contained in $\mathcal{W}_g$. Hence also their sum is contained in $\mathcal{W}_g$.

($\subset$) Assume that $x_0 \in \mathcal{W}_g$. Let u be a Bohl input such that $Hx_u(\cdot, x_0)$ is stable. Denote $x := x_u(\cdot, x_0)$. Obviously, the input u and the state trajectory x can be decomposed uniquely as

$$x = x_1 + x_2 \text{ and } u = u_1 + u_2,$$

with x_1, x_2, u_1 and u_2 Bohl, the spectrum of u_1 and x_1 contained in $\mathbb{C}_g$ and the spectrum of u_2 and x_2 contained in $\mathbb{C}_b$. Denote $x_{10} := x_1(0)$ and $x_{20} := x_2(0)$. Then we have $x_0 = x_{10} + x_{20}$. Also, since $\dot{x} = Ax + Bu$, we have

$$\dot{x}_1(t) - Ax_1(t) - Bu_1(t) = -\dot{x}_2(t) + Ax_2(t) + Bu_2(t).$$

Note that in this equation the left hand side has its spectrum contained in $\mathbb{C}_g$, whereas the right hand side has its spectrum contained in $\mathbb{C}_b$ (see (2.7)). It follows that both sides of the equation must in fact be identically equal to zero. Hence we obtain

$$\dot{x}_1(t) = Ax_1(t) + Bu_1(t), \tag{4.29}$$

$$\dot{x}_2(t) = Ax_2(t) + Bu_2(t). \tag{4.30}$$

From (4.29) it follows that $x_1 = x_{u_1}(\cdot, x_{10})$. Since x_1 is stable, according to (4.22) we have $x_{10} \in \mathcal{X}_{\text{stab}}$. On the other hand,

$$Hx_2 = Hx - Hx_1.$$

Since $\sigma(x_2) \subset \mathbb{C}_b$ we have that the spectrum of Hx_2 is contained in $\mathbb{C}_b$. However, both Hx as well as Hx_1 are stable so $\sigma(Hx_2) = \sigma(Hx - Hx_1) \subset \mathbb{C}_g$. This implies that $Hx_2(t) = 0$ for all t. It follows from (4.30) that $x_2 = x_{u_2}(\cdot, x_{20})$ and hence, by definition 4.4, that $x_{20} \in \mathcal{V}^*(\ker H)$. Thus $x_0 = x_{10} + x_{20} \in \mathcal{X}_{\text{stab}} + \mathcal{V}^*(\ker H)$. ∎

It follows from the above that $\mathcal{W}_g$ is a *strongly invariant subspace* (see exercise 4.2). Indeed, by combining theorem 4.37 and theorem 4.26 it is easy to see that $\mathcal{W}_g$ is A-invariant and that $\operatorname{im} B \subset \mathcal{W}_g$. Hence, $A\mathcal{W}_g + \operatorname{im} B \subset \mathcal{W}_g$. In particular this implies that $(A + BF)\mathcal{W}_g + \operatorname{im} B \subset \mathcal{W}_g$ for *any* linear map $F : \mathcal{X} \to \mathcal{U}$.

As already noted, the subspace inclusion $\operatorname{im} E \subset \mathcal{W}_g$ is a necessary condition for the existence of a map $F : \mathcal{X} \to \mathcal{U}$ such that G_F is stable . Using the representation for $\mathcal{W}_g$ obtained in theorem 4.37 we now prove that this subspace inclusion is also sufficient.

Lemma 4.38 *There exists a linear map* $F \in \underline{F}(\mathcal{V}^*(\ker H))$ *such that*

$$\sigma(A_F \mid \mathcal{W}_g/\mathcal{V}^*(\ker H)) \subset \mathbb{C}_g.$$

Proof : Recall that $\mathcal{W}_g$ is A-invariant and that $\operatorname{im} B \subset \mathcal{W}_g$. Let $A_0 := A \mid \mathcal{W}_g$, the restriction of A to $\mathcal{W}_g$, and consider the system $\dot{x} = A_0x + Bu$ with state space $\mathcal{W}_g$. Denote $\mathcal{V}^* := \mathcal{V}^*(\ker H)$. We have $\mathcal{V}^* \subset \mathcal{W}_g$, and its is easily seen that $\mathcal{V}^*$ is controlled invariant with respect to the restricted system (A_0, B). Also, $\mathcal{X}_{\text{stab}} \subset \mathcal{W}_g$ and it can be verified that the stabilizable subspace of (A_0, B) is equal to $\mathcal{X}_{\text{stab}}$ (use the characterization (4.22)). Consequently, the formula

$$\mathcal{V}^* + \mathcal{X}_{\text{stab}} = \mathcal{W}_g,$$

together with theorem 4.26 implies that $\mathcal{V}^*$ is outer-stabilizable with respect to the system (A_0, B). Hence there exists an $F : \mathcal{W}_g \to \mathcal{U}$ such that $(A_0 + BF)\mathcal{V}^* \subset \mathcal{V}^*$ and

$$\sigma(A_0 + BF \mid \mathcal{W}_g/\mathcal{V}^*) \subset \mathbb{C}_g.$$

Extend this F to a map on $\mathcal{X}$ in an arbitrary way. Since A_0 and A coincide on $\mathcal{W}_g$, we obtain $\sigma(A_F \mid \mathcal{W}_g/\mathcal{V}^*) \subset \mathbb{C}_g$. ■

Theorem 4.39 *Consider the system (4.25). There exists a linear map $F : \mathcal{X} \to \mathcal{U}$ such that $G_F(s)$ is stable if and only if* $\operatorname{im} E \subset \mathcal{V}^*(\ker H) + \mathcal{X}_{\text{stab}}$.

Proof : (⇒) This was already proven. (⇐) This is an application of lemma 4.38: Let F be such that $A_F\mathcal{V}^* \subset \mathcal{V}^*$ and $\sigma(A_F \mid \mathcal{W}_g/\mathcal{V}^*) \subset \mathbb{C}_g$ ($\mathcal{W}_g$ is automatically A_F-invariant). Since $\mathcal{V}^* \subset \mathcal{W}_g$, $\mathcal{V}^* \subset \ker H$ and $\operatorname{im} E \subset \mathcal{W}_g$, we may conclude from lemma 4.35 that $G_F(s)$ is stable. ■

By using theorem 4.26, we see that the subspace inclusion of theorem 4.39 can in principle be verified computationally. Indeed, $\mathcal{X}_{\text{stab}} = \mathcal{X}_g(A) + \langle A \mid \operatorname{im} B\rangle$ so $\mathcal{X}_{\text{stab}}$ can be calculated from first principles. In section 4.3 we gave an algorithm to compute $\mathcal{V}^*(\ker H)$. Of course, again we do not address numerical issues here.

4.9 Exercises

4.1 (Output null-controllability.) Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad z(t) = Hx(t).$$

If $x_0 \in \mathcal{X}$ and u is an input function, then the corresponding output is denoted by $z_u(t, x_0) := Hx_u(t, x_0)$. A point x_0 is called *output null-controllable* if there is a $T > 0$ and an input u such that $z_u(t, x_0) = 0$ for all $t \geqslant T$. The subspace of all output null-controllable points is denoted by $\mathcal{S}$. Prove that

$$\mathcal{S} = \mathcal{V}^*(\ker H) + \langle A \mid \operatorname{im} B\rangle.$$

Hint: $x_u(T, x_0) = e^{AT}x_0 + x_u(T, 0)$. Use the facts that

$$x_u(T, x_0) \in \mathcal{V}^*(\ker H),$$
$$x_u(T, 0) \in \langle A \mid \operatorname{im} B\rangle$$

and that

$$\mathcal{V}^*(\ker H) + \langle A \mid \operatorname{im} B\rangle$$

is A-invariant.

4.2 (Strong invariance.) Consider the system $\dot{x}(t) = Ax(t) + Bu(t)$. A subspace $\mathcal{V}$ of $\mathcal{X}$ is called *strongly invariant* if for each $x_0 \in \mathcal{V}$ and for each input function u we have $x_u(t, x_0) \in \mathcal{V}$ for all $t \geqslant 0$. Show that

a. The reachable subspace $\langle A \mid \operatorname{im} B\rangle$ is strongly invariant.

b. $\mathcal{V}$ is strongly invariant if and only if $\mathcal{V}$ is controlled invariant and $\langle A \mid \operatorname{im} B\rangle \subset \mathcal{V}$.

c. If $\mathcal{V}$ is strongly invariant then $(A + BF)\mathcal{V} \subset \mathcal{V}$ for all F.

d. $\mathcal{V}$ is strongly invariant if and only if $A\mathcal{V} + \operatorname{im} B \subset \mathcal{V}$.

4.3 Consider the system (A, B). Let $F : \mathcal{X} \to \mathcal{U}$ be a linear map and let $G : \mathcal{U} \to \mathcal{U}$ be an isomorphism. Show that the classes of (A, B)-invariant subspaces and $(A + BF, BG)$-invariant subspaces coincide.

4.4 Consider the system $\dot{x}(t) = Ax(t) + Bu(t)$. Let $x_0 \in \mathcal{X}$, let u be an input function, and let $x_u(\cdot, x_0)$ be the resulting state trajectory. Let $\mathcal{V}$ denote the linear span of the vectors $\{x_u(t, x_0) \mid t \geqslant 0\}$. Show that $\mathcal{V}$ is a controlled invariant subspace.

4.5 (The model matching problem) In this exercise we study the connection between DDP and the solvability of a rational matrix equation. Consider the system (4.5). Define $R_1(s) := H(Is - A)^{-1}B$ and $R_2(s) := H(Is - A)^{-1}E$. R_1 and R_2 are strictly proper real rational matrices of dimensions $q \times m$ and $q \times r$, respectively. We consider the equation

$$R_1 Q = R_2$$

in the unknown Q, which is required to be a $(m \times r)$ strictly proper real rational matrix. A more systemic interpretation of this equation is the following: given a system Σ_1 with transfer matrix $R_1(s)$ and a system Σ_2 with transfer matrix $R_2(s)$, find a system Σ_m such that the cascade (= parallel) connection of Σ_m and Σ_1 is equal to Σ_2

The above is called the problem of *exact model matching*: given the system Σ_1 (plant) and a 'desired' system Σ_2, find a 'precompensator' Σ_m for Σ_1 such that the resulting cascade connection has exactly the same input/output behaviour as the given system Σ_2.

a. Show that the equation $R_1 Q = R_2$ has a strictly proper real rational solution Q if and only if there exists an $(m \times r)$ matrix Bohl function U such that

$$\int_0^t He^{A(t-\tau)} BU(\tau)\, \mathrm{d}\tau = He^{At} \text{ for all } t \geqslant 0.$$

b. Show that if there exists U such that the equation in (i) holds, then for each $x_0 \in \operatorname{im} E$ there exists an input function u for the system $\dot{x}(t) = Ax(t) + Bu(t)$ such that $Hx_u(t, x_0) = 0$ for all $t \geqslant 0$.

c. Show that if $F \in \underline{F}(\mathcal{V}^*(\ker H))$ then for each $x_0 \in \mathcal{V}^*(\ker H)$ we have

$$-R_1(s)F(Is - A - BF)^{-1}x_0 = H(Is - A)^{-1}x_0.$$

d. Conclude that the equation $R_1 Q = R_2$ has a strictly proper real rational solution Q if and only if $\operatorname{im} E \subset \mathcal{V}^*(\ker H)$.

4.6 (Disturbance decoupling with feedforward.) Consider the system (4.5). In the previous section we studied the problem of disturbance decoupling by state feedback. Sometimes, instead of restricting ourselves to feedback control laws of the form $u(t) = Fx(t)$, we want to allow the use of control laws of the form $u(t) = Fx(t) + Nd(t)$. If such a control law is connected to our system, then the closed-loop system is given by the equation

$$\dot{x}(t) = (A + BF)x(t) + (BN + E)d(t), \; z(t) = Hx(t).$$

Thus we may pose the problem of disturbance decoupling by state feedback *with feedforward*: find linear maps $F : \mathcal{X} \to \mathcal{U}$ and $N : \mathcal{D} \to \mathcal{U}$ such that the given closed-loop system is disturbance decoupled.

a. Show that the closed-loop system is disturbance decoupled if and only if there exists an $(A + BF)$-invariant subspace $\mathcal{V}$ such that $\operatorname{im}(BN + E) \subset \mathcal{V} \subset \ker H$.

b. Let $N : \mathcal{D} \to \mathcal{U}$ be given. Show that there exists $F : \mathcal{X} \to \mathcal{U}$ such that the closed-loop system is disturbance decoupled if and only if $\operatorname{im}(BN + E) \subset \mathcal{V}^*(\ker H)$.

c. Show that there exist F and N such that the closed-loop system is disturbance decoupled if and only if $\operatorname{im} E \subset \mathcal{V}^*(\ker H) + \operatorname{im} B$.

4.7 Consider the system $\dot{x}(t) = Ax(t) + Bu(t)$. A subspace $\mathcal{V}$ of $\mathcal{X}$ is called a *reachability subspace* if for every $x_1 \in \mathcal{V}$ there exists $T > 0$ and an input function u such that $x_u(t, 0) \in \mathcal{V}$ for all $0 \leqslant t \leqslant T$ and $x_u(T, 0) = x_1$, i.e., if every point in the subspace can be reached from the origin in finite time along a trajectory that does not leave the subspace. Show that:

a. every reachability subspace is controlled invariant,

b. a subspace $\mathcal{V}$ is a reachability subspace if and only if it is a controllability subspace,

c. a subspace $\mathcal{V}$ is a controllability subspace if and only if it has the property that for any pair of points $x_0, x_1 \in \mathcal{V}$ there exists $T > 0$ and an input function u such that $x_u(t, x_0) \in \mathcal{V}$ for all $0 \leqslant t \leqslant T$ and $x_u(T, x_0) = x_1$.

4.8 Consider the system $\dot{x}(t) = Ax(t) + Bu(t)$. A subspace $\mathcal{V}$ of $\mathcal{X}$ is called a *coasting subspace* if for each $x_0 \in \mathcal{V}$ there is *exactly one* input function u such that $x_u(t, x_0) \in \mathcal{V}$ for all $t \geqslant 0$. Show that the following three conditions are equivalent:

a. $\mathcal{V}$ is a coasting subspace,

b. $\mathcal{V}$ is controlled invariant, $\mathcal{R}^*(\mathcal{V}) = 0$ and B is injective,

c. $\underline{F}(\mathcal{V}) \neq \varnothing$ and if $F_1, F_2 \in \underline{F}(\mathcal{V})$ then $F_1 \mid \mathcal{V} = F_2 \mid \mathcal{V}$.

4.9 Consider the single-input system $\dot{x}(t) = Ax(t) + bu(t)$. Assume that (A, b) is controllable.

a. Find all controllability subspaces associated with the system (A, b).

b. Show that every controlled invariant subspace $\mathcal{V}$ with $\mathcal{V} \neq \mathcal{X}$ is a coasting subspace.

Let $\mathcal{K}$ be a subspace of $\mathcal{X}$ with $\mathcal{K} \neq \mathcal{X}$. Assume that $x_0 \in \mathcal{K}$ and let u be such that $x_u(t, x_0) \in \mathcal{K}$ for all $t \geqslant 0$.

c. Show that u is given by a state feedback control law, i.e., there is a linear map $f : \mathcal{X} \to \mathcal{U}$ such that $u = fx$.

4.10 (Output regulation by state feedback.) Consider the system $\dot{x}(t) = Ax(t) + Bu(t) + Ed(t)$, $z(t) = Hx(t)$. For a given feedback control law $u(t) = Fx(t)$, let $z_F(t, x_0, d)$ denote the output of the closed-loop system corresponding to the initial condition x_0 and disturbance d. In this exercise we study the problem of *output regulation by state feedback*. We will say that F *achieves output regulation* if $z_F(t, x_0, d) \to 0$ $(t \to \infty)$ for all $x_0 \in \mathcal{X}$ and every disturbance d.

a. Show that F achieves output regulation if and only if $He^{A_F t}E = 0$ for all t and $He^{A_F t} \to 0$ $(t \to \infty)$.

b. Let $\mathcal{X}_{\text{stab}}$ denote the stabilizable subspace of the pair (A, B) with respect to the stability set $\mathbb{C}^- = \{s \in \mathbb{C} \mid \Re\text{e}\, s < 0\}$. Show that there exists F such that $He^{A_F t} \to 0$ $(t \to \infty)$ if and only if $\mathcal{X} = \mathcal{V}^*(\ker H) + \mathcal{X}_{\text{stab}}$.

c. Show that there exists a map F that achieves output regulation if and only if $\mathcal{V}^*(\ker H)$ is outer-stabilizable and $\operatorname{im} E \subset \mathcal{V}^*(\ker H)$.

4.11 (Input/output stabilization with feedforward.) Again consider the system (4.5). Suppose that $\mathbb{C}_g$ is a stability domain. The problem of input/output stabilization by state feedback *with feedforward* is to find a control law $u(t) = Fx(t) + Nd(t)$ such that the transfer function of the resulting closed-loop system, i.e.

$$G_{F,N}(s) := H(Is - A_F)^{-1}(BN + E),$$

is stable (see also exercise 4.6). Show that there exists a control law $u(t) = Fx(t) + Nd(t)$ such that $G_{F,N}(s)$ is stable if and only if $\operatorname{im} E \subset \mathcal{V}^*(\ker H) + \mathcal{X}_{\text{stab}}$. Conclude that allowing feedforward of the disturbance input does not enlarge the class of systems that can be made input/output stable.

4.12 Give a proof of theorem 4.25.

4.13 (Disturbance decoupling by state feedback with pole placement.) In addition to the ordinary disturbance decoupling problem, DDP, and the disturbance decoupling problem with stability, DDPS, we can also consider the disturbance decoupling problem with pole placement, DDPPP. Here, the question is to find conditions under which for *any* stability domain $\mathbb{C}_g$, there exists a map $F : \mathcal{X} \to \mathcal{U}$ such that $\sigma(A + BF) \subset \mathbb{C}_g$ and $T_F = 0$ (where, as usual, T_F denotes the closed loop impulse response from d to z). In this exercise we derive necessary and sufficient conditions for this to hold. Denote $\mathcal{V}^*(\mathcal{K})$ by $\mathcal{V}^*$, $\mathcal{R}^*(\mathcal{K})$ by $\mathcal{R}^*$, and $\mathcal{V}_g^*(\mathcal{K})$ by $\mathcal{V}_g^*$.

- **a.** Observe that if for any stability domain $\mathbb{C}_g$ there exists a map $F : \mathcal{X} \to \mathcal{U}$ such that $\sigma(A+BF) \subset \mathbb{C}_g$ and $T_F = 0$, then for any stability domain $\mathbb{C}_g$ we have $\operatorname{im} E \subset \mathcal{V}_g^*$.
- **b.** For $F \in \underline{F}(\mathcal{V}^*)$, let τ denote the fixed spectrum $\sigma(A + BF \mid \mathcal{V}^*/\mathcal{R}^*)$. Show that if $\mathbb{C}_g$ is a stability domain with the property that $\tau \cap \mathbb{C}_g = \varnothing$, then $\mathcal{V}_g^* = \mathcal{R}^*$.
- **c.** Show that if (A, B) is controllable, then for any pair of real monic polynomials (p_1, p_2) such that $\deg p_1 = \dim \mathcal{R}^*$ and $\deg p_2 = n - \dim \mathcal{R}^*$, there exist $F \in \underline{F}(\mathcal{R}^*)$ such that $\chi_{A_F|\mathcal{R}^*} = p_1$ and $\chi_{A_F|\mathcal{X}/\mathcal{R}^*} = p_2$. (Hint: apply theorem 4.18 to $\mathcal{V} = \mathcal{R}^*$).
- **d.** Show that if (A, B) is controllable and $\operatorname{im} E \subset \mathcal{R}^*$, then for any real monic polynomial p of degree n such that $p = p_1 p_2$, with p_1 and p_2 monic polynomials and $\deg p_1 = \dim \mathcal{R}^*$, there exists $F : \mathcal{X} \to \mathcal{U}$ such that $\chi_{A_F} = p$ and $T_F = 0$.
- **e.** Show that for any stability domain $\mathbb{C}_g$ there exists a map $F : \mathcal{X} \to \mathcal{U}$ such that $\sigma(A + BF) \subset \mathbb{C}_g$ and $T_F = 0$ if and only if (A, B) is controllable and $\operatorname{im} E \subset \mathcal{R}^*$.

4.10 Notes and references

Controlled invariant subspaces were introduced independently by Basile and Marro [10,11] and Wonham and Morse [224]. An extensive treatment, including the disturbance decoupling problem by state feedback, can also be found in Wonham's classical textbook [223], and in the textbook [14] by Basile and Marro. A characterization of controlled invariant subspaces in terms of vectors of rational functions was given by Hautus in [72]. A study of controlled invariant subspaces in the context of polynomial models can be found in the work of Fuhrmann and Willems [51].

Alternative conditions for the existence of disturbance decoupling state feedback control laws, in terms of the open loop control input-to-output, and disturbance input-to-output transfer matrices, were obtained by Bhattacharyya in [19]. Robustness issues in the context of design of disturbance decoupling state feedback controllers were studied by Bhattacharyya, Del Nero Gomez and Howze in [22], and by Bhattacharyya in [21]. Extensions to characterize the freedom in placing the closed loop