7 Differential forms

Exterior differential forms arise when concepts such as the work of a field along a path and the flux of a fluid through a surface are generalized to higher dimensions.

Hamiltonian mechanics cannot be understood without differential forms. The information we need about differential forms involves exterior multiplication, exterior differentiation, integration, and Stokes' formula.

32 Exterior forms

Here we define exterior algebraic forms

A *1-forms*

Let $\mathbb{R}^n$ be an n-dimensional real vector space.[52] We will denote vectors in this space by $\boldsymbol{\xi}, \boldsymbol{\eta}, \ldots$.

Definition. A form of degree 1 (or a 1-form) is a linear function $\omega\colon \mathbb{R}^n \to \mathbb{R}$, i.e.,

$$\omega(\lambda_1 \boldsymbol{\xi}_1 + \lambda_2 \boldsymbol{\xi}_2) = \lambda_1 \omega(\boldsymbol{\xi}_1) + \lambda_2 \omega(\boldsymbol{\xi}_2), \qquad \lambda_1, \lambda_2 \in \mathbb{R} \text{ and } \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^n.$$

We recall the basic facts about 1-forms from linear algebra. The set of all 1-forms becomes a real vector space if we define the sum of two forms by

$$(\omega_1 + \omega_2)\boldsymbol{\xi} = \omega_1(\boldsymbol{\xi}) + \omega_2(\boldsymbol{\xi}),$$

and scalar multiplication by

$$(\lambda\omega)(\boldsymbol{\xi}) = \lambda\omega(\boldsymbol{\xi}).$$

[52] It is essential to note that we do not fix any special euclidean structure on $\mathbb{R}^n$. In some *examples* we use such a structure; in these cases this will be specifically stated ("euclidean $\mathbb{R}^n$").

The space of 1-forms on $\mathbb{R}^n$ is itself n-dimensional, and is also called the dual space $(\mathbb{R}^n)^*$.

Suppose that we have chosen a linear coordinate system $x_1, \ldots, x_n$ on $\mathbb{R}^n$. Each coordinate x_i is itself a 1-form. These n 1-forms are linearly independent. Therefore, every 1-form ω has the form

$$\omega = a_1 x_1 + \cdots + a_n x_n, \qquad a_i \in \mathbb{R}.$$

The value of ω on a vector ξ is equal to

$$\omega(\xi) = a_1 x_1(\xi) + \cdots + a_n x_n(\xi),$$

where $x_1(\xi), \ldots, x_n(\xi)$ are the components of ξ in the chosen coordinate system.

Example. If a uniform free field $\mathbf{F}$ is given on euclidean $\mathbb{R}^3$, its work A on the displacement ξ is a 1-form acting on ξ (Figure 135).

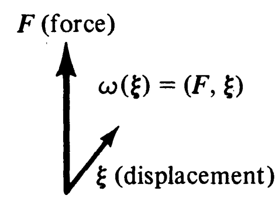

Figure 135 The work of a force is a 1-form acting on the displacement.

B *2-forms*

Definition. An exterior form of degree 2 (or a 2-form) is a function on pairs of vectors $\omega^2: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, which is bilinear and skew symmetric:

$$\omega^2(\lambda_1 \xi_1 + \lambda_2 \xi_2, \xi_3) = \lambda_1 \omega^2(\xi_1, \xi_3) + \lambda_2 \omega^2(\xi_2, \xi_3)$$
$$\omega^2(\xi_1, \xi_2) = -\omega^2(\xi_2, \xi_1),$$
$$\forall \lambda_1, \lambda_2 \in \mathbb{R}, \xi_1, \xi_2, \xi_3 \in \mathbb{R}^n.$$

Example 1. Let $S(\xi_1, \xi_2)$ be the oriented area of the parallelogram constructed on the vectors ξ_1 and ξ_2 of the oriented euclidean plane $\mathbb{R}^2$, i.e.,

$$S(\xi_1, \xi_2) = \begin{vmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{vmatrix}, \quad \text{where } \xi_1 = \xi_{11}\mathbf{e}_1 + \xi_{12}\mathbf{e}_2, \xi_2 = \xi_{21}\mathbf{e}_1 + \xi_{22}\mathbf{e}_2,$$

with $\mathbf{e}_1, \mathbf{e}_2$ a basis giving the orientation on $\mathbb{R}^2$.

It is easy to see that $S(\xi_1, \xi_2)$ is a 2-form (Figure 136).

Example 2. Let $\mathbf{v}$ be a uniform velocity vector field for a fluid in three-dimensional oriented euclidean space (Figure 137). Then the flux of the fluid over the area of the parallelogram ξ_1, ξ_2 is a bilinear skew symmetric function of ξ_1 and ξ_2, i.e., a 2-form defined by the triple scalar product

$$\omega^2(\xi_1\ \xi_2) = (\mathbf{v}, \xi_1, \xi_2).$$

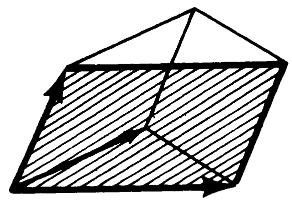

Figure 136 Oriented area is a 2-form.

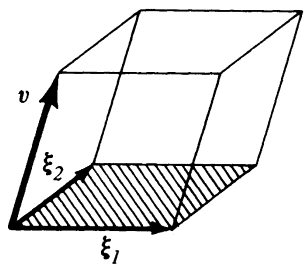

Figure 137 Flux of a fluid through a surface is a 2-form.

EXAMPLE 3. The oriented area of the projection of the parallelogram with sides ξ_1 and ξ_2 on the x_1, x_2-plane in euclidean $\mathbb{R}^3$ is a 2-form.

PROBLEM 1. Show that for every 2-form ω^2 on $\mathbb{R}^n$ we have

$$\omega^2(\xi, \xi) = 0, \qquad \forall \xi \in \mathbb{R}^n.$$

Solution. By skew symmetry, $\omega^2(\xi, \xi) = -\omega^2(\xi, \xi)$.

The set of all 2-forms on $\mathbb{R}^n$ becomes a real vector space if we define the addition of forms by the formula

$$(\omega_1 + \omega_2)(\xi_1, \xi_2) = \omega_1(\xi_1, \xi_2) + \omega_2(\xi_1, \xi_2)$$

and multiplication by scalars by the formula

$$(\lambda\omega)(\xi_1, \xi_2) = \lambda\omega(\xi_1, \xi_2).$$

PROBLEM 2. Show that this space is finite-dimensional, and find its dimension.
ANSWER. $n(n-1)/2$; a basis is shown below.

C *k-forms*

Definition. An exterior form of degree k, or a k-form, is a function of k vectors which is k-linear and antisymmetric:

$$\omega(\lambda_1\xi_1' + \lambda_2\xi_1'', \xi_2, \ldots, \xi_k) = \lambda_1\omega(\xi_1', \xi_2, \ldots, \xi_k) + \lambda_2\omega(\xi_1'', \xi_2, \ldots, \xi_k)$$

$$\omega(\xi_{i_1}, \ldots, \xi_{i_k}) = (-1)^\nu\omega(\xi_1, \ldots, \xi_k),$$

where

$$\nu = \begin{cases} 0 & \text{if the permutation } i_1, \ldots, i_k \text{ is even;} \\ 1 & \text{if the permutation } i_1, \ldots, i_k \text{ is odd.} \end{cases}$$

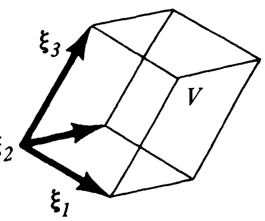

Figure 138 Oriented volume is a 3-form.

EXAMPLE 1. The oriented volume of the parallelepiped with edges $\xi_1, \ldots, \xi_n$ in oriented euclidean space $\mathbb{R}^n$ is an n-form (Figure 138).

$$V(\xi_1, \ldots, \xi_n) = \begin{vmatrix} \xi_{11} & \cdots & \xi_{1n} \\ \vdots & & \vdots \\ \xi_{n1} & \cdots & \xi_{nn} \end{vmatrix},$$

where $\xi_i = \xi_{i1}\mathbf{e}_1 + \cdots + \xi_{in}\mathbf{e}_n$ and $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are a basis of $\mathbb{R}^n$.

EXAMPLE 2. Let $\mathbb{R}^k$ be an oriented k-plane in n-dimensional euclidean space $\mathbb{R}^n$. Then the k-dimensional oriented volume of the projection of the parallelepiped with edges $\xi_1, \xi_2, \ldots, \xi_k \in \mathbb{R}^n$ onto $\mathbb{R}^k$ is a k-form on $\mathbb{R}^n$.

The set of all k-forms in $\mathbb{R}^n$ form a real vector space if we introduce operations of addition

$$(\omega_1 + \omega_2)\xi = \omega_1(\xi) + \omega_2(\xi), \qquad \xi = \{\xi_1, \ldots, \xi_k\}, \xi_j \in \mathbb{R}^n,$$

and multiplication by scalars

$$(\lambda\omega)(\xi) = \lambda\omega(\xi).$$

PROBLEM 3. Show that this vector space is finite-dimensional and find its dimension.
ANSWER. C_n^k; a basis is shown below.

D *The exterior product of two 1-forms*

We now introduce one more operation: exterior multiplication of forms. If ω^k is a k-form and ω^l is an l-form on $\mathbb{R}^n$, then their exterior product $\omega^k \wedge \omega^l$ will be a $k + l$-form. We first define the exterior product of 1-forms, which associates to every pair of 1-forms ω_1, ω_2 on $\mathbb{R}^n$ a 2-form $\omega_1 \wedge \omega_2$ on $\mathbb{R}^n$.

Let ξ be a vector in $\mathbb{R}^n$. Given two 1-forms ω_1 and ω_2, we can define a mapping of $\mathbb{R}^n$ to the plane $\mathbb{R} \times \mathbb{R}$ by associating to $\xi \in \mathbb{R}^n$ the vector $\omega(\xi)$ with components $\omega_1(\xi)$ and $\omega_2(\xi)$ in the plane with coordinates ω_1, ω_2 (Figure 139).

Definition. The value of the exterior product $\omega_1 \wedge \omega_2$ on the pair of vectors $\xi_1, \xi_2 \in \mathbb{R}^n$ is the oriented area of the image of the parallelogram with sides $\omega(\xi_1)$ and $\omega(\xi_2)$ on the ω_1, ω_2-plane:

$$(\omega_1 \wedge \omega_2)(\xi_1, \xi_2) = \begin{vmatrix} \omega_1(\xi_1) & \omega_2(\xi_1) \\ \omega_1(\xi_2) & \omega_2(\xi_2) \end{vmatrix}.$$

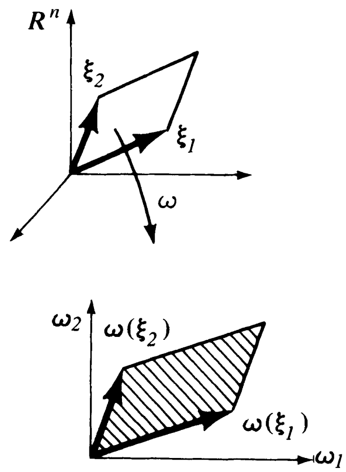

Figure 139 Definition of the exterior product of two 1-forms

PROBLEM 4. Show that $\omega_1 \wedge \omega_2$ really is a 2-form.

PROBLEM 5. Show that the mapping

$$(\omega_1, \omega_2) \to \omega_1 \wedge \omega_2$$

is bilinear and skew symmetric:

$$\omega_1 \wedge \omega_2 = -\omega_2 \wedge \omega_1,$$

$$(\lambda'\omega_1' + \lambda''\omega_1'') \wedge \omega_2 = \lambda'\omega_1' \wedge \omega_2 + \lambda''\omega_1'' \wedge \omega_2.$$

Hint. The determinant is bilinear and skew-symmetric not only with respect to rows, but also with respect to columns.

Now suppose we have chosen a system of linear coordinates on $\mathbb{R}^n$, i.e., we are given n independent 1-forms $x_1, \ldots, x_n$. We will call these forms *basic*.

The exterior products of the basic forms are the 2-forms $x_i \wedge x_j$. By skew-symmetry, $x_i \wedge x_i = 0$ and $x_i \wedge x_j = -x_j \wedge x_i$. The geometric meaning of the form $x_i \wedge x_j$ is very simple: its value on the pair of vectors ξ_1, ξ_2 is equal to the oriented area of the image of the parallelogram ξ_1, ξ_2 on the coordinate plane x_i, x_j under the projection parallel to the remaining coordinate directions.

PROBLEM 6. Show that the $C_n^2 = n(n-1)/2$ forms $x_i \wedge x_j (i < j)$ are linearly independent.

In particular, in three-dimensional euclidean space (x_1, x_2, x_3), the area of the projection on the (x_1, x_2)-plane is $x_1 \wedge x_2$, on the (x_2, x_3)-plane it is $x_2 \wedge x_3$, and on the (x_3, x_1)-plane it is $x_3 \wedge x_1$.

PROBLEM 7. Show that every 2-form in the three-dimensional space (x_1, x_2, x_3) is of the form

$$Px_2 \wedge x_3 + Qx_3 \wedge x_1 + Rx_1 \wedge x_2.$$

PROBLEM 8. Show that every 2-form on the n-dimensional space with coordinates $x_1, \dots, x_n$ can be uniquely represented in the form

$$\omega^2 = \sum_{i<j} a_{ij} x_i \wedge x_j.$$

Hint. Let $\mathbf{e}_i$ be the i-th basis vector, i.e., $x_i(\mathbf{e}_i) = 1$, $x_j(\mathbf{e}_i) = 0$ for $i \neq j$. Look at the value of the form ω^2 on the pair $\mathbf{e}_i, \mathbf{e}_j$. Then

$$a_{ij} = \omega^2(\mathbf{e}_i, \mathbf{e}_j).$$

E *Exterior monomials*

Suppose that we are given k 1-forms $\omega_1, \dots, \omega_k$. We define their exterior product $\omega_1 \wedge \cdots \wedge \omega_k$.

Definition. Set

$$(\omega_1 \wedge \cdots \wedge \omega_k)(\xi_1, \dots, \xi_k) = \begin{vmatrix} \omega_1(\xi_1) & \cdots & \omega_k(\xi_1) \\ \vdots & & \vdots \\ \omega_1(\xi_k) & \cdots & \omega_k(\xi_k) \end{vmatrix}.$$

In other words, the value of a product of 1-forms on the parallelepiped $\xi_1, \dots, \xi_k$ is equal to the oriented volume of the image of the parallelepiped in the oriented euclidean coordinate space $\mathbb{R}^k$ under the mapping $\xi \to (\omega_1(\xi), \dots, \omega_k(\xi))$.

PROBLEM 9. Show that $\omega_1 \wedge \cdots \wedge \omega_k$ is a k-form.

PROBLEM 10. Show that the operation of exterior product of 1-forms gives a multi-linear skew-symmetric mapping

$$(\omega_1, \dots, \omega_k) \to \omega_1 \wedge \dots \wedge \omega_k.$$

In other words,

$$(\lambda'\omega_1' + \lambda''\omega_1'') \wedge \omega_2 \wedge \cdots \wedge \omega_k = \lambda'\omega_1' \wedge \omega_2 \wedge \cdots \wedge \omega_k + \lambda''\omega_1'' \wedge \omega_2 \wedge \cdots \wedge \omega_k$$

and

$$\omega_{i_1} \wedge \cdots \wedge \omega_{i_k} = (-1)^{\nu} \omega_1 \wedge \cdots \wedge \omega_k,$$

where

$$\nu = \begin{cases} 0 & \text{if the permutation } i_1, \dots, i_k \text{ is even,} \\ 1 & \text{if the permutation } i_1, \dots, i_k \text{ is odd.} \end{cases}$$

Now consider a coordinate system on $\mathbb{R}^n$ given by the basic forms $x_1, \dots, x_n$. The exterior product of k basic forms

$$x_{i_1} \wedge \cdots \wedge x_{i_k}, \qquad 1 \leq i_m \leq n,$$

is the oriented volume of the image of a k-parallelepiped on the k-plane $(x_{i_1}, \dots, x_{i_k})$ under the projection parallel to the remaining coordinate directions.

PROBLEM 11. Show that, if two of the indices $i_1, \ldots, i_k$ are the same, then the form $x_{i_1} \wedge \cdots \wedge x_{i_k}$ is zero.

PROBLEM 12. Show that the forms

$$x_{i_1} \wedge \cdots \wedge x_{i_k}, \quad \text{where } 1 \leq i_1 < i_2 < \cdots < i_k \leq n,$$

are linearly independent.

The number of such forms is clearly C_n^k. We will call them *basic k-forms*.

PROBLEM 13. Show that every k-form on $\mathbb{R}^n$ can be uniquely represented as a linear combination of basic forms:

$$\omega^k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1, \ldots, i_k} x_{i_1} \wedge \cdots \wedge x_{i_k}.$$

Hint. $a_{i_1, \ldots, i_k} = \omega^k(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k})$.

It follows as a result of this problem that the dimension of the vector space of k-forms on $\mathbb{R}^n$ is equal to C_n^k. In particular, for $k = n$, $C_n^k = 1$, from which follows

Corollary. *Every n-form on $\mathbb{R}^n$ is either the oriented volume of a parallelepiped with some choice of unit volume, or zero:*

$$\omega^n = a \cdot x_1 \wedge \cdots \wedge x_n.$$

PROBLEM 14. Show that every k-form on $\mathbb{R}^n$ with $k > n$ is zero.

We now consider the product of a k-form ω^k and an l-form ω^l. First, suppose that we are given two monomials

$$\omega^k = \omega_1 \wedge \cdots \wedge \omega_k \quad \text{and} \quad \omega^l = \omega_{k+1} \wedge \cdots \wedge \omega_{k+l},$$

where $\omega_1, \ldots, \omega_{k+l}$ are 1-forms. We define their product $\omega^k \wedge \omega^l$ to be the monomial

$$(\omega_1 \wedge \cdots \wedge \omega_k) \wedge (\omega_{k+1} \wedge \cdots \wedge \omega_{k+l}) = \omega_1 \wedge \cdots \wedge \omega_k \wedge \omega_{k+1} \wedge \cdots \wedge \omega_{k+l}.$$

PROBLEM 15. Show that the product of monomials is associative:

$$(\omega^k \wedge \omega^l) \wedge \omega^m = \omega^k \wedge (\omega^l \wedge \omega^m)$$

and skew-commutative:

$$\omega^k \wedge \omega^l = (-1)^{kl} \omega^l \wedge \omega^k.$$

Hint. In order to move each of the l factors of ω^l forward, we need k inversions with the k factors of ω^k.

Remark. It is useful to remember that skew-commutativity means commutativity only if one of the degrees k and l is even, and anti-commutativity if both degrees k and l are odd.

33 Exterior multiplication

We define here the operation of exterior multiplication of forms and show that it is skew-commutative, distributive, and associative.

A *Definition of exterior multiplication*

We now define the exterior multiplication of an arbitrary k-form ω^k by an arbitrary l-form ω^l. The result $\omega^k \wedge \omega^l$ will be a $k + l$-form. The operation of multiplication turns out to be:

1. skew-commutative: $\omega^k \wedge \omega^l = (-1)^{kl}\omega^l \wedge \omega^k$;
2. distributive: $(\lambda_1\omega_1^k + \lambda_2\omega_2^k) \wedge \omega^l = \lambda_1\omega_1^k \wedge \omega^l + \lambda_2\omega_2^k \wedge \omega^l$;
3. associative: $(\omega^k \wedge \omega^l) \wedge \omega^m = \omega^k \wedge (\omega^l \wedge \omega^n)$.

Definition. The exterior product $\omega^k \wedge \omega^l$ of a k-form ω^k on $\mathbb{R}^n$ with an l-form ω^l on $\mathbb{R}^n$ is the $k + l$-form on $\mathbb{R}^n$ whose value on the $k + l$ vectors $\xi_1, \ldots, \xi_k, \xi_{k+1}, \ldots, \xi_{k+l} \in \mathbb{R}^n$ is equal to

(1) $\quad (\omega^k \wedge \omega^l)(\xi_1, \ldots, \xi_{k+1}) = \sum (-1)^\nu \omega^k(\xi_{i_1}, \ldots, \xi_{i_k})\omega^l(\xi_{j_1}, \ldots, \xi_{j_l}),$

where $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_l$; $(i_1, \ldots, i_k, j_1, \ldots, j_l)$ is a permutation of the numbers $(1, 2, \ldots, k + l)$; and

$$\nu = \begin{cases} 1 & \text{if this permutation is odd;} \\ 0 & \text{if this permutation is even.} \end{cases}$$

In other words, every partition of the $k + l$ vectors $\xi_1, \ldots, \xi_{k+l}$ into two groups (of k and of l vectors) gives one term in our sum (1). This term is equal to the product of the value of the k-form ω^k on the k vectors of the first group with the value of the l-form ω^l on the l vectors of the second group, with sign $+$ or $-$ depending on how the vectors are ordered in the groups. If they are ordered in such a way that the k vectors of the first group and the l vectors of the second group written in succession form an even permutation of the vectors $\xi_1, \xi_2, \ldots, \xi_{k+l}$, then we take the sign to be $+$, and if they form an odd permutation we take the sign to be $-$.

Example. If $k = l = 1$, then there are just two partitions: ξ_1, ξ_2 and ξ_2, ξ_1. Therefore,

$$(\omega_1 \wedge \omega_2)(\xi_1, \xi_2) = \omega_1(\xi_1)\omega_2(\xi_2) - \omega_2(\xi_1)\omega_1(\xi_2),$$

which agrees with the definition of multiplication of 1-forms in Section 32.

Problem 1. Show that the definition above actually defines a $k + l$-form (i.e., that the value of $(\omega^k \wedge \omega^l)(\xi_1, \ldots, \xi_{k+l})$ depends linearly and skew-symmetrically on the vectors ξ).

B *Properties of the exterior product*

Theorem. *The exterior multiplication of forms defined above is skew-commutative, distributive, and associative. For monomials it coincides with the multiplication defined in Section* 32.

The proof of skew-commutativity is based on the simplest properties of even and odd permutations (cf. the problem at the end of Section 32) and will be left to the reader.

Distributivity follows from the fact that every term in (1) is linear with respect to ω^k and ω^l.

The proof of associativity requires a little more combinatorics. Since the corresponding arguments are customarily carried out in algebra courses for the proof of Laplace's theorem on the expansion of a determinant by column minors, we may use this theorem.[53]

We begin with the following observation: if associativity is proved for the terms of a sum, then it is also true for the sum, i.e.,

$$\left.\begin{aligned}(\omega_1' \wedge \omega_2) \wedge \omega_3 = \omega_1' \wedge (\omega_2 \wedge \omega_3)\\ (\omega_1'' \wedge \omega_2) \wedge \omega_3 = \omega_1'' \wedge (\omega_2 \wedge \omega_3)\end{aligned}\right\} \text{ implies}$$
$$((\omega_1' + \omega_1'') \wedge \omega_2) \wedge \omega_3 = (\omega_1' + \omega_1'') \wedge (\omega_2 \wedge \omega_3).$$

But, by distributivity, which has already been proved, we have

$$((\omega_1' + \omega_1'') \wedge \omega_2) \wedge \omega_3 = ((\omega_1' \wedge \omega_2) \wedge \omega_3) + ((\omega_1'' \wedge \omega_2) \wedge \omega_3),$$
$$(\omega_1' + \omega_1'') \wedge (\omega_2 \wedge \omega_3) = (\omega_1' \wedge (\omega_2 \wedge \omega_3)) + (\omega_1'' \wedge (\omega_2 \wedge \omega_3)).$$

We already know from Section 32 (Problem 12) that every form on $\mathbb{R}^n$ is a sum of monomials; therefore, it is enough to show associativity for multiplication of monomials.

Since we have not yet proved the equivalence of the definition in Section 32 of multiplication of k 1-forms with the general definition (1), we will temporarily denote the multiplication of k 1-forms by the symbol $\bar{\wedge}$, so that our monomials have the form

$$\omega^k = \omega_1 \mathbin{\bar{\wedge}} \cdots \mathbin{\bar{\wedge}} \omega_k \quad \text{and} \quad \omega^l = \omega_{k+1} \mathbin{\bar{\wedge}} \cdots \mathbin{\bar{\wedge}} \omega_{k+l},$$

where $\omega_1, \ldots, \omega_{k+l}$ are 1-forms.

[53] A direct proof of associativity (also containing a proof of Laplace's theorem) consists of checking the signs in the identity

$$((\omega^k \wedge \omega^l) \wedge \omega^m)(\xi_1, \ldots, \xi_{k+l+m}) = \sum \pm\, \omega^k(\xi_{i_1}, \ldots, \xi_{i_k})\omega^l(\xi_{j_1}, \ldots, \xi_{j_l})\omega^m(\xi_{h_1}, \ldots \xi_{h_m}),$$

where $i_1 < \cdots < i_k, j_1 < \cdots < j_l, h_1 < \cdots < h_m; (i_1, \ldots, h_m)$ is a permutation of the numbers $(1, \ldots, k + l + m)$.

Lemma. *The exterior product of two monomials is a monomial:*

$$(\omega_1 \barwedge \cdots \barwedge \omega_k) \wedge (\omega_{k+1} \barwedge \cdots \barwedge \omega_{k+l})$$
$$= \omega_1 \barwedge \cdots \barwedge \omega_k \barwedge \omega_{k+1} \barwedge \cdots \barwedge \omega_{k+l}.$$

PROOF. We calculate the values of the left and right sides on $k + l$ vectors $\xi_1, \ldots, \xi_{k+l}$. The value of the left side, by formula (1), is equal to the sum of the products

$$\sum \pm \det_{1 \le i \le k} |\omega_i(\xi_{i_m})| \cdot \det_{k < i \le k+l} |\omega_i(\xi_{j_m})|$$

of the minors of the first k columns of the determinant of order $k + l$ and the remaining minors. Laplace's theorem on the expansion by minors of the first k columns asserts exactly that this sum, with the same rule of sign choice as in Definition (1), is equal to the determinant $\det|\omega_i(\xi_j)|$. □

It follows from the lemma that the operations $\barwedge$ and $\wedge$ coincide: we get, in turn,

$$\omega_1 \barwedge \omega_2 = \omega_1 \wedge \omega_2,$$
$$\omega_1 \barwedge \omega_2 \barwedge \omega_3 = (\omega_1 \barwedge \omega_2) \wedge \omega_3 = (\omega_1 \wedge \omega_2) \wedge \omega_3,$$
$$\omega_1 \barwedge \omega_2 \barwedge \cdots \barwedge \omega_k = (\cdots((\omega_1 \wedge \omega_2) \wedge \omega_3) \wedge \cdots \wedge \omega_k).$$

The associativity of $\wedge$-multiplication of monomials therefore follows from the obvious associativity of $\barwedge$-multiplication of 1-forms. Thus, in view of the observation made above, associativity is proved in the general case.

PROBLEM 2. Show that the exterior square of a 1-form, or, in general, of a form of odd order, is equal to zero: $\omega^k \wedge \omega^k = 0$ if k is odd.

EXAMPLE 1. Consider a coordinate system $p_1, \ldots, p_n, q_1, \ldots, q_n$ on $\mathbb{R}^{2n}$ and the 2-form $\omega^2 = \sum_{i=1}^n p_i \wedge q_i$.

[Geometrically, this form signifies the sum of the oriented areas of the projection of a parallelogram on the n two-dimensional coordinate planes $(p_1, q_1), \ldots, (p_n, q_n)$. Later, we will see that the 2-form ω^2 has a special meaning for hamiltonian mechanics. It can be shown that every nondegenerate[54] 2-form on $\mathbb{R}^{2n}$ has the form ω^2 in some coordinate system $(p_1, \ldots, q_n)$.]

PROBLEM 3. Find the exterior square of the 2-form ω^2.

ANSWER.

$$\omega^2 \wedge \omega^2 = -2\sum_{i>j} p_i \wedge p_j \wedge q_i \wedge q_j.$$

PROBLEM 4. Find the exterior k-th power of ω^2.

ANSWER.

$$\underbrace{\omega^2 \wedge \omega^2 \wedge \cdots \wedge \omega^2}_{k} = \pm k! \sum_{i_1 < \cdots < i_k} p_{i_1} \wedge \cdots \wedge p_{i_k} \wedge q_{i_1} \wedge \cdots \wedge q_{i_k}.$$

[54] A bilinear form ω^2 is nondegenerate if $\forall \xi \neq 0, \exists \eta: \omega^2(\xi, \eta) \neq 0$.

In particular,

$$\underbrace{\omega^2 \wedge \cdots \wedge \omega^2}_{n} = \pm n! p_1 \wedge \cdots \wedge p_n \wedge q_1 \wedge \cdots \wedge q_n$$

is, up to a factor, the volume of a $2n$-dimensional parallelepiped in $\mathbb{R}^{2n}$.

EXAMPLE 2. Consider the oriented euclidean space $\mathbb{R}^3$. Every vector $\mathbf{A} \in \mathbb{R}^3$ determines a 1-form $\omega_{\mathbf{A}}^1$, by $\omega_{\mathbf{A}}^1(\xi) = (\mathbf{A}, \xi)$ (scalar product) and a 2-form $\omega_{\mathbf{A}}^2$ by

$$\omega_{\mathbf{A}}^2(\xi_1, \xi_2) = (\mathbf{A}, \xi_1, \xi_2) \qquad \text{(triple scalar product).}$$

PROBLEM 5. Show that the maps $\mathbf{A} \to \omega_{\mathbf{A}}^1$ and $\mathbf{A} \to \omega_{\mathbf{A}}^2$ establish isomorphisms of the linear space $\mathbb{R}^3$ of vectors $\mathbf{A}$ with the linear spaces of 1-forms on $\mathbb{R}^3$ and 2-forms on $\mathbb{R}^3$. If we choose an orthonormal oriented coordinate system (x_1, x_2, x_3) on $\mathbb{R}^3$, then

$$\omega_{\mathbf{A}}^1 = A_1 x_1 + A_2 x_2 + A_3 x_3$$

and

$$\omega_{\mathbf{A}}^2 = A_1 x_2 \wedge x_3 + A_2 x_3 \wedge x_1 + A_3 x_1 \wedge x_2.$$

Remark. Thus the isomorphisms do not depend on the choice of the orthonormal oriented coordinate system (x_1, x_2, x_3). But they do depend on the choice of the euclidean structure on $\mathbb{R}^3$, and the isomorphism $\mathbf{A} \to \omega_{\mathbf{A}}^2$ also depends on the orientation (coming implicitly in the definition of triple scalar product).

PROBLEM 6. Show that, under the isomorphisms established above, the exterior product of 1-forms becomes the vector product in $\mathbb{R}^3$, i.e., that

$$\omega_{\mathbf{A}}^1 \wedge \omega_{\mathbf{B}}^1 = \omega_{[\mathbf{A},\mathbf{B}]}^2 \quad \text{for any } \mathbf{A}, \mathbf{B} \in \mathbb{R}^3.$$

In this way the exterior product of 1-forms can be considered as an extension of the vector product in $\mathbb{R}^3$ to higher dimensions. However, in the n-dimensional case, the product is not a vector in the same space: the space of 2-forms on $\mathbb{R}^n$ is isomorphic to $\mathbb{R}^n$ only for $n = 3$.

PROBLEM 7. Show that, under the isomorphisms established above, the exterior product of a 1-form and a 2-form becomes the scalar product of vectors in $\mathbb{R}^3$:

$$\omega_{\mathbf{A}}^1 \wedge \omega_{\mathbf{B}}^2 = (\mathbf{A}, \mathbf{B}) x_1 \wedge x_2 \wedge x_3.$$

C *Behavior under mappings*

Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be a linear map, and ω^k an exterior k-form on $\mathbb{R}^n$. Then there is a k-form $f^*\omega^k$ on $\mathbb{R}^m$, whose value on the k vectors $\xi_1, \ldots, \xi_k \in \mathbb{R}^m$ is equal to the value of ω^k on their images:

$$(f^*\omega^k)(\xi_1, \ldots, \xi_k) = \omega^k(f\xi_1, \ldots, f\xi_k).$$

PROBLEM 8. Verify that $f^*\omega^k$ is an exterior form.

PROBLEM 9. Verify that f^* is a linear operator from the space of k-forms on $\mathbb{R}^n$ to the space of k-forms on $\mathbb{R}^m$ (the star *superscript* means that f^* acts in the opposite direction from f).

PROBLEM 10. Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ and $g\colon \mathbb{R}^n \to \mathbb{R}^p$. Verify that $(g \circ f)^* = f^* \circ g^*$.

PROBLEM 11. Verify that f^* preserves exterior multiplication: $f^*(\omega^k \wedge \omega^l) = (f^*\omega^k) \wedge (f^*\omega^l)$.

34 Differential forms

We give here the definition of differential forms on differentiable manifolds.

A *Differential 1-forms*

The simplest example of a differential form is the differential of a function.

EXAMPLE. Consider the function $y = f(x) = x^2$. Its differential $df = 2x\,dx$ depends on the point x and on the "increment of the argument," i.e., on the tangent vector ξ to the x axis. We fix the point x. Then the differential of the function at x, $df|_x$, depends *linearly on* ξ. So, if $x = 1$ and the coordinate of the tangent vector ξ is equal to 1, then $df = 2$, and if the coordinate of ξ is equal to 10, then $df = 20$ (Figure 140).

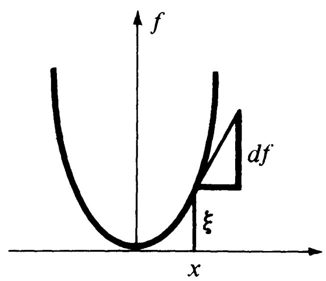

Figure 140 Differential of a function

Let $f: M \to \mathbb{R}$ be a differentiable function on the manifold M (we can imagine a "function of many variables" $f: \mathbb{R}^n \to \mathbb{R}$). The differential $df|_x$ of f at $\mathbf{x}$ is a linear map

$$df_{\mathbf{x}}: TM_{\mathbf{x}} \to \mathbb{R}$$

of the tangent space to M at $\mathbf{x}$ into the real line. We recall from Section 18C the definition of this map:

Let $\xi \in TM_{\mathbf{x}}$ be the velocity vector of the curve $\mathbf{x}(t): \mathbb{R} \to M$; $\mathbf{x}(0) = \mathbf{x}$ and $\dot{\mathbf{x}}(0) = \xi$. Then, by definition,

$$df_{\mathbf{x}}(\xi) = \left.\frac{d}{dt}\right|_{t=0} f(\mathbf{x}(t)).$$

PROBLEM 1. Let ξ be the velocity vector of the plane curve $x(t) = \cos t$, $y(t) = \sin t$ at $t = 0$. Calculate the values of the differentials dx and dy of the functions x and y on the vector ξ (Figure 141).

ANSWER. $\quad dx|_{(1,0)}(\xi) = 0,\ dy|_{(1,0)}(\xi) = 1$

Note that the differential of a function f at a point $\mathbf{x} \in M$ is a 1-form $df_{\mathbf{x}}$ on the tangent space $TM_{\mathbf{x}}$.

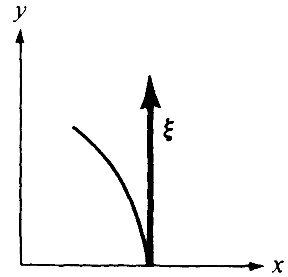

Figure 141 Problem 1

The differential df of f on the manifold M is a smooth map of the tangent bundle TM to the line

$$df: TM \to \mathbb{R} \qquad \left(TM = \bigcup_{\mathbf{x}} TM_{\mathbf{x}}\right).$$

This map is differentiable and is linear on each tangent space $TM_{\mathbf{x}} \subset TM$.

Definition. *A differential form of degree* 1 (*or a* 1*-form*) *on a manifold M* is a smooth map

$$\omega: TM \to \mathbb{R}$$

of the tangent bundle of M to the line, linear on each tangent space $TM_{\mathbf{x}}$.

One could say that a *differential* 1*-form on M is an algebraic* 1*-form on* $TM_{\mathbf{x}}$ *which is "differentiable with respect to* $\mathbf{x}$."

PROBLEM 2. Show that every differential 1-form on the line is the differential of some function.

PROBLEM 3. Find differential 1-forms on the circle and the plane which are not the differential of any function.

B *The general form of a differential 1-form on* $\mathbb{R}^n$

We take as our manifold M a vector space with coordinates $x_1, \ldots, x_n$. Recall that the components $\xi_1, \ldots, \xi_n$ of a tangent vector $\boldsymbol{\xi} \in T\mathbb{R}^n_{\mathbf{x}}$ are the values of the differentials $dx_1, \ldots, dx_n$ on the vector $\boldsymbol{\xi}$. These n 1-forms on $T\mathbb{R}^n_{\mathbf{x}}$ are linearly independent. Thus the 1-forms $dx_1, \ldots, dx_n$ form a basis for the n-dimensional space of 1-forms on $T\mathbb{R}^n_{\mathbf{x}}$, and every 1-form on $T\mathbb{R}^n_{\mathbf{x}}$ can be uniquely written in the form $a_1\, dx_1 + \cdots + a_n\, dx_n$, where the a_i are real coefficients. Now let ω be an arbitrary differential 1-form on $\mathbb{R}^n$. At every point $\mathbf{x}$ it can be expanded uniquely in the basis $dx_1, \ldots, dx_n$. From this we get:

Theorem. *Every differential* 1*-form on the space* $\mathbb{R}^n$ *with a given coordinate system* $x_1, \ldots, x_n$ *can be written uniquely in the form*

$$\omega = a_1(x)dx_1 + \cdots + a_n(x)dx_n,$$

where the coefficients $a_i(x)$ *are smooth functions.*

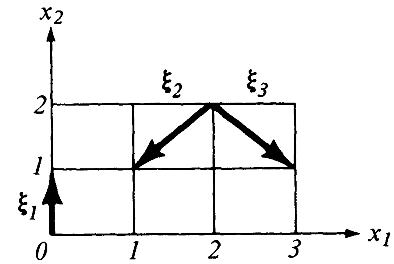

Figure 142 Problem 4

PROBLEM 4. Calculate the value of the forms $\omega_1 = dx_1$, $\omega_2 = x_1 dx_2$, and $\omega_3 = dr^2$ ($r^2 = x_1^2 + x_2^2$) on the vectors ξ_1, ξ_2, and ξ_3 (Figure 142).

ANSWER.

	ξ_1	ξ_2	ξ_3
ω_1	0	-1	1
ω_2	0	-2	-2
ω_3	0	-8	0

PROBLEM 5. Let $x_1, \ldots, x_n$ be functions on a manifold M forming a local coordinate system in some region. Show that every 1-form on this region can be uniquely written in the form $\omega = a_1(x)\, dx_1 + \cdots + a_n(x)\, dx_n$.

C *Differential k-forms*

Definition. *A differential k-form $\omega^k|_{\mathbf{x}}$ at a point $\mathbf{x}$ of a manifold M is an exterior k-form on the tangent space $TM_{\mathbf{x}}$ to M at $\mathbf{x}$, i.e., a k-linear skew-symmetric function of k vectors $\xi_1, \ldots, \xi_k$ tangent to M at $\mathbf{x}$.*

If such a form $\omega^k|_{\mathbf{x}}$ is given at every point $\mathbf{x}$ of the manifold M and if it is differentiable, then we say that we are given a *k-form ω^k on the manifold M*.

PROBLEM 6. Put a natural differentiable manifold structure on the set whose elements are k-tuples of vectors tangent to M at some point $\mathbf{x}$.

A differential k-form is a smooth map from the manifold of Problem 6 to the line.

PROBLEM 7. Show that the k-forms on M form a vector space (infinite-dimensional if k does not exceed the dimension of M).

Differential forms can be multiplied by functions as well as by numbers. Therefore, the set of C^∞ differential k-forms has a natural structure as a module over the ring of infinitely differentiable real functions on M.

D *The general form of a differential k-form on* $\mathbb{R}^n$

Take as the manifold M the vector space $\mathbb{R}^n$ with fixed coordinate functions $x_1, \ldots, x_n: \mathbb{R}^n \to \mathbb{R}$. Fix a point $\mathbf{x}$. We saw above that the n 1-forms $dx_1, \ldots, dx_n$ form a basis of the space of 1-forms on the tangent space $T\mathbb{R}^n_{\mathbf{x}}$.

Consider exterior products of the basic forms:

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \qquad i_1 < \cdots < i_k.$$

In Section 32 we saw that these C_n^k k-forms form a basis of the space of exterior k-forms on $T\mathbb{R}^n_{\mathbf{x}}$. Therefore, every exterior k-form on $T\mathbb{R}^n_{\mathbf{x}}$ can be written uniquely in the form

$$\sum_{i_1 < \cdots < i_k} a_{i_1, \ldots, i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Now let ω be an arbitrary differential k-form on $\mathbb{R}^n$. At every point $\mathbf{x}$ it can be uniquely expressed in terms of the basis above. From this follows:

Theorem. *Every differential k-form on the space* $\mathbb{R}^n$ *with a given coordinate system* $x_1, \ldots, x_n$ *can be written uniquely in the form*

$$\omega^k = \sum_{i_1 < \cdots < i_k} a_{i_1, \ldots, i_k}(\mathbf{x}) dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where the $a_{i_1, \ldots, i_k}(\mathbf{x})$ *are smooth functions on* $\mathbb{R}^n$.

PROBLEM 8. Calculate the value of the forms $\omega_1 = dx_1 \wedge dx_2$, $\omega_2 = x_1\, dx_1 \wedge dx_2 - x_2\, dx_2 \wedge dx_1$, and $\omega_3 = r\, dr \wedge d\varphi$ (where $x_1 = r\cos\varphi$ and $x_2 = r\sin\varphi$) on the pairs of vectors $(\boldsymbol{\xi}_1, \boldsymbol{\eta}_1)$, $(\boldsymbol{\xi}_2, \boldsymbol{\eta}_2)$, and $(\boldsymbol{\xi}_3, \boldsymbol{\eta}_3)$ (Figure 143).

ANSWER.

	$(\boldsymbol{\xi}_1, \boldsymbol{\eta}_1)$	$(\boldsymbol{\xi}_2, \boldsymbol{\eta}_2)$	$(\boldsymbol{\xi}_3, \boldsymbol{\eta}_3)$
ω_1	1	1	-1
ω_2	2	1	-3
ω_3	1	1	-1

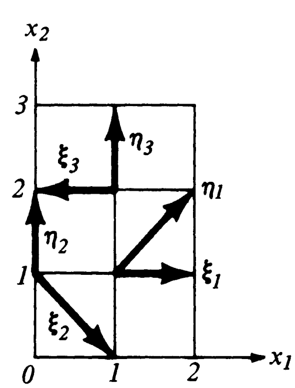

Figure 143 Problem 8

PROBLEM 9. Calculate the value of the forms $\omega_1 = dx_2 \wedge dx_3$, $\omega_2 = x_1\, dx_3 \wedge dx_2$, and $\omega_3 = dx_3 \wedge dr^2$ ($r^2 = x_1^2 + x_2^2 + x_3^2$), on the pair of vectors $\boldsymbol{\xi} = (1, 1, 1)$, $\boldsymbol{\eta} = (1, 2, 3)$ at the point $\mathbf{x} = (2, 0, 0)$.

ANSWER. $\omega_1 = 1, \omega_2 = -2, \omega_3 = -8$.

PROBLEM 10. Let $x_1, \ldots, x_n \colon M \to \mathbb{R}$ be functions on a manifold which form a local coordinate system on some region. Show that every differential form on this region can be written uniquely in the form

$$\omega^k = \sum_{i_1 < \cdots < i_k} a_{i_1, \ldots, i_k}(\mathbf{x})\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

EXAMPLE. Change of variables in a form. Suppose that we are given two coordinate systems on $\mathbb{R}^3$: x_1, x_2, x_3 and y_1, y_2, y_3. Let ω be a 2-form on $\mathbb{R}^3$. Then, by the theorem above, ω can be written in the system of x-coordinates as $\omega = X_1\, dx_2 \wedge dx_3 + X_2\, dx_3 \wedge dx_1 + X_3\, dx_1 \wedge dx_2$, where X_1, X_2, and X_3 are functions of x_1, x_2, and x_3, and in the system of y-coordinates as $\omega = Y_1\, dy_2 \wedge dy_3 + Y_2\, dy_3 \wedge dy_1 + Y_3\, dy_1 \wedge dy_2$, where Y_1, Y_2, and Y_3 are functions of y_1, y_2, and y_3.

PROBLEM 11. Given the form written in the x-coordinates (i.e., the X_i) and the change of variables formulas $\mathbf{x} = \mathbf{x}(\mathbf{y})$, write the form in y-coordinates, i.e., find Y.

Solution. We have $dx_i = (\partial x_i/\partial y_1)\, dy_1 + (\partial x_i/\partial y_2)\, dy_2 + (\partial x_i/\partial y_3)\, dy_3$. Therefore,

$$dx_2 \wedge dx_3 = \left(\frac{\partial x_2}{\partial y_1} dy_1 + \frac{\partial x_2}{\partial y_2} dy_2 + \frac{\partial x_2}{\partial y_3} dy_3\right) \wedge \left(\frac{\partial x_3}{\partial y_1} dy_1 + \frac{\partial x_3}{\partial y_2} dy_2 + \frac{\partial x_3}{\partial y_3} dy_3\right),$$

from which we get

$$Y_3 = X_1 \left|\frac{D(x_2, x_3)}{D(y_1, y_2)}\right| + X_2 \left|\frac{D(x_3, x_1)}{D(y_1, y_2)}\right| + X_3 \left|\frac{D(x_1, x_2)}{D(y_1, y_2)}\right|, \text{ etc.}$$

E *Appendix. Differential forms in three-dimensional spaces*

Let M be a three-dimensional oriented riemannian manifold (in all future examples M will be euclidean three-space $\mathbb{R}^3$). Let x_1, x_2, and x_3 *be local coordinates, and let the square of the length element have the form*

$$ds^2 = E_1\, dx_1^2 + E_2\, dx_2^2 + E_3\, dx_3^2$$

(i.e., the coordinate system is triply orthogonal).

PROBLEM 12. Find E_1, E_2, and E_3 for cartesian coordinates x, y, z, for cylindrical coordinates r, φ, z and for spherical coordinates R, φ, θ in the euclidean space $\mathbb{R}^3$ (Figure 144).

ANSWER.

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2\, d\varphi^2 + dz^2 = dR^2 + R^2 \cos^2\theta\, d\varphi^2 + R^2\, d\theta^2.$$

We let $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ denote the unit vectors in the coordinate directions. These three vectors form a basis of the tangent space.

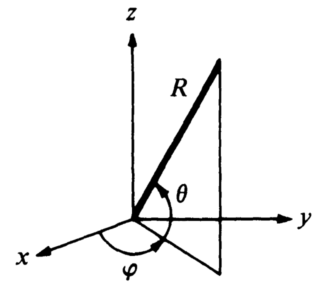

Figure 144 Problem 12

PROBLEM 13. Find the values of the forms dx_1, dx_2, and dx_3 on the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$.

ANSWER. $dx_i(\mathbf{e}_i) = 1/\sqrt{E_i}$, the rest are zero. In particular, for cartesian coordinates $dx(\mathbf{e}_x) = dy(\mathbf{e}_y) = dz(\mathbf{e}_z) = 1$; for cylindrical coordinates $dr(\mathbf{e}_r) = dz(\mathbf{e}_z) = 1$ and $d\varphi(\mathbf{e}_\varphi) = 1/r$ (Figure 145), for spherical coordinates $dR(\mathbf{e}_R) = 1$, $d\varphi(\mathbf{e}_\varphi) = 1/R \cos\theta$ and $d\theta(\mathbf{e}_\theta) = 1/R$.

The metric and orientation on the manifold M furnish the tangent space to M at every point with the structure of an oriented euclidean three-dimensional space. In terms of this structure, we can talk about scalar, vector, and triple scalar products.

PROBLEM 14. Calculate $[\mathbf{e}_1, \mathbf{e}_2]$, $(\mathbf{e}_R, \mathbf{e}_\theta)$, and $(\mathbf{e}_z, \mathbf{e}_x, \mathbf{e}_y)$.

ANSWER. $\mathbf{e}_3$, 0, 1.

In an oriented euclidean three-space every vector $\mathbf{A}$ corresponds to a 1-form $\omega_{\mathbf{A}}^1$ and a 2-form $\omega_{\mathbf{A}}^2$, defined by the conditions

$$\omega_{\mathbf{A}}^1(\boldsymbol{\xi}) = (\mathbf{A}, \boldsymbol{\xi}) \qquad \omega_{\mathbf{A}}^2(\boldsymbol{\xi}, \boldsymbol{\eta}) = (\mathbf{A}, \boldsymbol{\xi}, \boldsymbol{\eta}), \qquad \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^3.$$

The correspondence between vector fields and forms does not depend on the system of coordinates, but only on the euclidean structure and orientation. Therefore, every vector field $\mathbf{A}$ on our manifold M corresponds to a differential 1-form $\omega_{\mathbf{A}}^1$ on M and a differential 2-form $\omega_{\mathbf{A}}^2$ on M.

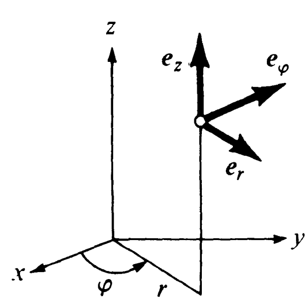

Figure 145 Problem 13

The formulas for changing from fields to forms and back have a different form in each coordinate system. Suppose that in the coordinates x_1, x_2, and x_3 described above, the vector field has the form

$$\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3$$

(the components A_i are smooth functions on M). The corresponding 1-form $\omega_{\mathbf{A}}^1$ decomposes over the basis dx_i, and the corresponding 2-form over the basis $dx_i \wedge dx_j$.

Problem 15. Given the components of the vector field $\mathbf{A}$, find the decompositions of the 1-form $\omega_{\mathbf{A}}^1$ and the 2-form $\omega_{\mathbf{A}}^2$.

Solution. We have $\omega_{\mathbf{A}}^1(\mathbf{e}_1) = (\mathbf{A}, \mathbf{e}_1) = A_1$. Also, $(a_1\, dx_1 + a_2\, dx_2 + a_3\, dx_3)(\mathbf{e}_1) = a_1\, dx_1(\mathbf{e}_1) = a_1/\sqrt{E_1}$. From this we get that $a_1 = A_1\sqrt{E_1}$, so that

$$\omega_{\mathbf{A}}^1 = A_1\sqrt{E_1}\, dx_1 + A_2\sqrt{E_2}\, dx_2 + A_3\sqrt{E_3}\, dx_3.$$

In the same way, we have $\omega_{\mathbf{A}}^2 = (\mathbf{A}, \mathbf{e}_2, \mathbf{e}_3) = A_1$. Also,

$$(\alpha_1\, dx_2 \wedge dx_3 + \alpha_2\, dx_3 \wedge dx_1 + \alpha_3\, dx_1 \wedge dx_2)(\mathbf{e}_2, \mathbf{e}_3) = \alpha_1 \frac{1}{\sqrt{E_2 E_3}}.$$

Hence, $\alpha_1 = A_1\sqrt{E_2 E_3}$, i.e.,

$$\omega_{\mathbf{A}}^2 = A_1\sqrt{E_2E_3}\, dx_2 \wedge dx_3 + A_2\sqrt{E_3E_1}\, dx_3 \wedge dx_1 + A_3\sqrt{E_1E_2}\, dx_1 \wedge dx_2.$$

In particular, in cartesian, cylindrical, and spherical coordinates on $\mathbb{R}^3$ the vector field

$$\mathbf{A} = A_x\mathbf{e}_x + A_y\mathbf{e}_y + A_z\mathbf{e}_z = A_r\mathbf{e}_r + A_\varphi\mathbf{e}_\varphi + A_z\mathbf{e}_z = A_R\mathbf{e}_R + A_\varphi\mathbf{e}_\varphi + A_\theta\mathbf{e}_\theta$$

corresponds to the 1-form

$$\omega_{\mathbf{A}}^1 = A_x\, dx + A_y\, dy + A_z\, dz = A_r\, dr + rA_\varphi\, d\varphi + A_z\, dz = A_R\, dR + R\cos\theta A_\varphi\, d\varphi + RA_\theta\, d\theta$$

and the 2-form

$$\begin{aligned}\omega_{\mathbf{A}}^2 &= A_x\, dy \wedge dz + A_y\, dz \wedge dx + A_z\, dx \wedge dy \\ &= rA_r\, d\varphi \wedge dz + A\, dz \wedge dr + rA_z\, dr \wedge d\varphi \\ &= R^2\cos\theta A_R\, d\varphi \wedge d\theta + RA_\varphi\, d\theta \wedge dR + R\cos\theta A_z\, dR \wedge d\varphi.\end{aligned}$$

An example of a vector field on a manifold M is the gradient of a function $f: M \to \mathbb{R}$. Recall that the gradient of a function is the vector field $\mathbf{grad}\, f$ corresponding to the differential:

$$\omega_{\mathbf{grad}\, f}^1 = df, \quad \text{i.e.,} \quad df(\xi) = (\mathbf{grad}\, f, \xi) \qquad \forall \xi.$$

Problem 16. Find the components of the gradient of a function in the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Solution. We have $df = (\partial f/\partial x_1)\, dx_1 + (\partial f/\partial x_2)\, dx_2 + (\partial f/\partial x_3)\, dx_3$. By the problem above

$$\mathbf{grad}\, f = \frac{1}{\sqrt{E_1}}\frac{\partial f}{\partial x_1}\mathbf{e}_1 + \frac{1}{\sqrt{E_3}}\frac{\partial f}{\partial x_2}\mathbf{e}_2 + \frac{1}{\sqrt{E_3}}\frac{\partial f}{\partial x_3}\mathbf{e}_3.$$

In particular, in cartesian, cylindrical, and spherical coordinates

$$\mathbf{grad}\, f = \frac{\partial f}{\partial x}\mathbf{e}_x + \frac{\partial f}{\partial y}\mathbf{e}_y + \frac{\partial f}{\partial z}\mathbf{e}_z = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \varphi}\mathbf{e}_\varphi + \frac{\partial f}{\partial z}\mathbf{e}_z$$

$$= \frac{\partial f}{\partial R}\mathbf{e}_R + \frac{1}{R\cos\theta}\frac{\partial f}{\partial \varphi}\mathbf{e}_\varphi + \frac{1}{R}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta.$$

35 Integration of differential forms

We define here the concepts of a chain, the boundary of a chain, and the integration of a form over a chain.

The integral of a differential form is a higher-dimensional generalization of such ideas as the flux of a fluid across a surface or the work of a force along a path.

A *The integral of a 1-form along a path*

We begin by integrating a 1-form ω^1 on a manifold M. Let

$$\gamma\colon [0 \le t \le 1] \to M$$

be a smooth map (the "path of integration"). The integral of the form ω^1 on the path γ is defined as a limit of Riemann sums. Every Riemann sum consists of the values of the form ω^1 on some tangent vectors $\boldsymbol{\xi}_i$ (Figure 146):

$$\int_\gamma \omega^1 = \lim_{\Delta \to 0} \sum_{i=1}^{n} \omega^1(\boldsymbol{\xi}_i).$$

The tangent vectors $\boldsymbol{\xi}_i$ are constructed in the following way. The interval $0 \le t \le 1$ is divided into parts $\Delta_i\colon t_i \le t \le t_{i+1}$ by the points t_i. The interval Δ_i can be looked at as a tangent vector $\boldsymbol{\Delta}_i$ to the t axis at the point t_i. Its image in the tangent space to M at the point $\gamma(t_i)$ is

$$\boldsymbol{\xi}_i = d\gamma|_{t_i}(\boldsymbol{\Delta}_i) \in TM_{\gamma(t_i)}.$$

The sum has a limit as the largest of the intervals Δ_i tends to zero. It is called the integral of the 1-form ω^1 along the path γ.

The definition of the integral of a k-form along a k-dimensional surface follows an analogous pattern. The surface of integration is partitioned into

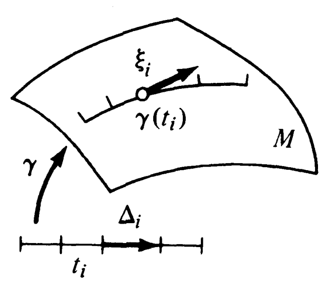

Figure 146 Integrating a 1-form along a path

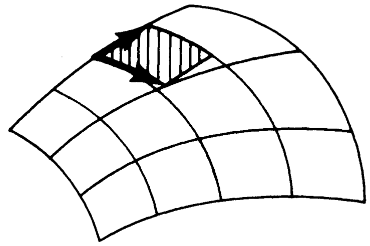

Figure 147 Integrating a 2-form over a surface

small curvilinear k-dimensional parallelepipeds (Figure 147); these parallelepipeds are replaced by parallelepipeds in the tangent space. The sum of the values of the form on the parallelepipeds in the tangent space approaches the integral as the partition is refined. We will first consider a particular case.

B *The integral of a k-form on oriented euclidean space* $\mathbb{R}^k$

Let $x_1, \ldots, x_k$ be an oriented coordinate system on $\mathbb{R}^k$. Then every k-form on $\mathbb{R}^k$ is proportional to the form $dx_1 \wedge \cdots \wedge dx_k$, i.e., it has the form $\omega^k = \varphi(x)dx_1 \wedge \cdots \wedge dx_k$, where $\varphi(x)$ is a smooth function.

Let D be a bounded convex polyhedron in $\mathbb{R}^k$ (Figure 148). By definition, the integral of the form ω^k on D is the integral of the function φ:

$$\int_D \omega^k = \int_D \varphi(x)dx_1, \ldots, dx_k,$$

where the integral on the right is understood to be the usual limit of Riemann sums.

Such a definition follows the pattern outlined above, since in this case the tangent space to the manifold is identified with the manifold.

Problem 1. Show that $\int_D \omega^k$ depends linearly on ω^k.

Problem 2. Show that if we divide D into two distinct polyhedra D_1 and D_2, then

$$\int_D \omega^k = \int_{D_1} \omega^k + \int_{D_2} \omega^k.$$

In the general case (a k-form on an n-dimensional space) it is not so easy to identify the elements of the partition with tangent parallelepipeds; we will consider this case below.

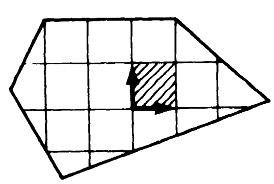

Figure 148 Integrating a k-form in k-dimensional space

C *The behavior of differential forms under maps*

Let $f: M \to N$ be a differentiable map of a smooth manifold M to a smooth manifold N, and let ω be a differential k-form on N (Figure 149). Then, a well-defined k-form arises also on M: it is denoted by $f^*\omega$ and is defined by the relation

$$(f^*\omega)(\xi_1, \ldots, \xi_k) = \omega(f_*\xi_1, \ldots, f_*\xi_k)$$

for any tangent vectors $\xi_1, \ldots, \xi_k \in TM_x$. Here f_* is the differential of the map f. In other words, the value of the form $f^*\omega$ on the vectors $\xi_1, \ldots, \xi_k$ is equal to the value of ω on the images of these vectors.

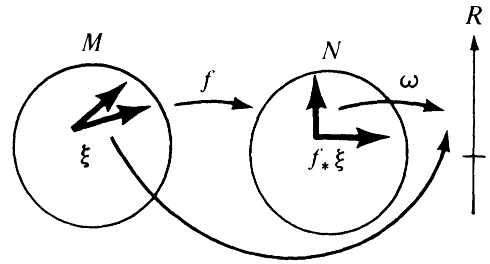

Figure 149 A form on N induces a form on M.

Example. If $y = f(x_1, x_2) = x_1^2 + x_2^2$ and $\omega = dy$, then

$$f^*\omega = 2x_1\,dx_1 + 2x_2\,dx_2.$$

Problem 3. Show that $f^*\omega$ is a k-form on M.

Problem 4. Show that the map f^* preserves operations on forms:

$$f^*(\lambda_1\omega_1 + \lambda_2\omega_2) = \lambda_1 f^*(\omega_1) + \lambda_2 f^*(\omega_2),$$
$$f^*(\omega_1 \wedge \omega_2) = (f^*\omega_1) \wedge (f^*\omega_2).$$

Problem 5. Let $g: L \to M$ be a differentiable map. Show that $(fg)^* = g^*f^*$.

Problem 6. Let D_1 and D_2 be two compact, convex polyhedra in the oriented k-dimensional space $\mathbb{R}^k$ and $f: D_1 \to D_2$ a differentiable map which is an orientation-preserving diffeomorphism[55] of the interior of D_1 onto the interior of D_2. Then, for any differential k-form ω^k on D_2,

$$\int_{D_1} f^*\omega^k = \int_{D_2} \omega^k.$$

Hint. This is the change of variables theorem for a multiple integral:

$$\int_{D_1} \frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)} \varphi(y(x))dx_1 \cdots dx_n = \int_{D_2} \varphi(y)dy_1 \cdots dy_n.$$

[55] i.e., one-to-one with a differentiable inverse.

D *Integration of a k-form on an n-dimensional manifold*

Let ω be a differential k-form on an n-dimensional manifold M. Let D be a bounded convex k-dimensional polyhedron in k-dimensional euclidean space $\mathbb{R}^k$ (Figure 150). The role of "path of integration" will be played by a

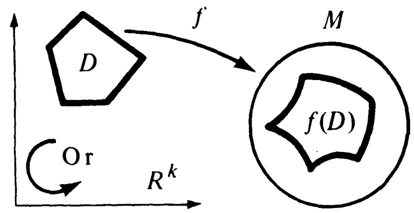

Figure 150 Singular k-dimensional polyhedron

k-dimensional cell[56] σ of M represented by a triple $\omega = (D, f, \mathrm{Or})$ consisting of

1. a convex polyhedron $D \subset \mathbb{R}^k$,
2. a differentiable map $f: D \to M$, and
3. an orientation on $\mathbb{R}^k$, denoted by Or.

Definition. The integral of the k-form ω over the k-dimensional cell σ is the integral of the corresponding form over the polyhedron D

$$\int_\sigma \omega = \int_D f^*\omega.$$

PROBLEM 7. Show that the integral depends linearly on the form:

$$\int_\sigma \lambda_1\omega_1 + \lambda_2\omega_2 = \lambda_1 \int_\sigma \omega_1 + \lambda_2 \int_\sigma \omega_2.$$

The k-dimensional cell which differs from σ only by the choice of orientation is called the *negative of* σ and is denoted by $-\sigma$ or $-1 \cdot \sigma$ (Figure 151).

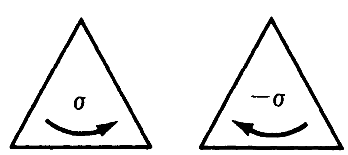

Figure 151 Problem 8

PROBLEM 8. Show that, under a change of orientation, the integral changes sign:

$$\int_{-\sigma} \omega = -\int_\sigma \omega.$$

[56] The cell σ is usually called a *singular k-dimensional polyhedron*.

E *Chains*

The set $f(D)$ is not necessarily a smooth submanifold of M. It could have "self-intersections" or "folds" and could even be reduced to a point. However, even in the one-dimensional case, it is clear that it is inconvenient to restrict ourselves to contours of integration consisting of one piece: it is useful to be able to consider contours consisting of several pieces which can be traversed in either direction, perhaps more than once. The analogous concept in higher dimensions is called a *chain.*

Definition. *A chain of dimension n on a manifold M* consists of a finite collection of n-dimensional oriented cells $\sigma_1, \ldots, \sigma_r$ in M and integers $m_1, \ldots, m_r$, called *multiplicities* (the multiplicities can be positive, negative, or zero). A chain is denoted by

$$c_k = m_1\sigma_1 + \cdots + m_r\sigma_r.$$

We introduce the natural identifications

$$m_1\sigma + m_2\sigma = (m_1 + m_2)\sigma$$

$$m_1\sigma_1 + m_2\sigma_2 = m_2\sigma_2 + m_1\sigma_1 \qquad 0\sigma = 0 \qquad c_k + 0 = c_k.$$

PROBLEM 9. Show that the set of all k-chains on M forms a commutative group if we define the addition of chains by the formula

$$(m_1\sigma_1 + \cdots + m_r\sigma_r) + (m_1'\sigma_1' + \cdots + m_{r_1}'\sigma_{r_1}') = m_1\sigma_1 + \cdots + m_r\sigma_r + m_1'\sigma_1' + \cdots + m_{r_1}'\sigma_{r_1}'.$$

F *Example: the boundary of a polyhedron*

Let D be a convex oriented k-dimensional polyhedron in k-dimensional euclidean space $\mathbb{R}^k$. The *boundary* of D is the $(k-1)$-chain ∂D on $\mathbb{R}^k$ defined in the following way (Figure 152).

The cells σ_i of the chain ∂D are the $(k-1)$-dimensional faces D_i of the polyhedron D, together with maps $f_i: D_i \to \mathbb{R}^k$ embedding the faces in $\mathbb{R}^k$ and orientations Or_i defined below; the multiplicities are equal to 1:

$$\partial D = \sum \sigma_i \qquad \sigma_i = (D_i, f_i, \mathrm{Or}_i).$$

Rule of orientation of the boundary. Let $\mathbf{e}_1, \ldots, \mathbf{e}_k$ be an oriented frame in $\mathbb{R}^k$. Let D_i be one of the faces of D. We choose an interior point of D_i and there

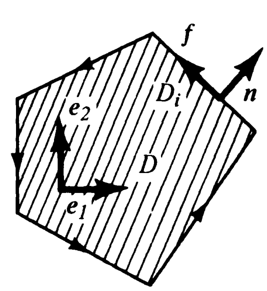

Figure 152 Oriented boundary

construct a vector $\mathbf{n}$ outwardly normal to the polyhedron D. An orienting frame for the face D_i will be a frame $\mathbf{f}_1, \ldots, \mathbf{f}_{k-1}$ on D_i such that the frame $(\mathbf{n}, \mathbf{f}_1, \ldots, \mathbf{f}_{k-1})$ is oriented *correctly* (i.e., the same way as the frame $\mathbf{e}_1, \ldots, \mathbf{e}_k$).

The *boundary of a chain* is defined in an analogous way. Let $\sigma = (D, f, \mathrm{Or})$ be a k-dimensional cell in the manifold M. Its boundary $\partial\sigma$ is the $(k-1)$ chain: $\partial\sigma = \sum \sigma_i$ consisting of the cells $\sigma_i = (D_i, f_i, \mathrm{Or}_i)$, where the D_i are the $(k-1)$-dimensional faces of D, Or_i are orientations chosen by the rule above, and f_i are the restrictions of the mapping $f: D \to M$ to the face D_i.

The boundary ∂c_k of the k-dimensional chain c_k in M is the sum of the boundaries of the cells of c_k with multiplicities (Figure 153):

$$\partial c_k = \partial(m_1\sigma_1 + \cdots + m_r\sigma_r) = m_1\,\partial\sigma_1 + \cdots + m_r\,\partial\sigma_r.$$

Obviously, ∂c_k is a $(k-1)$-chain on M.[57]

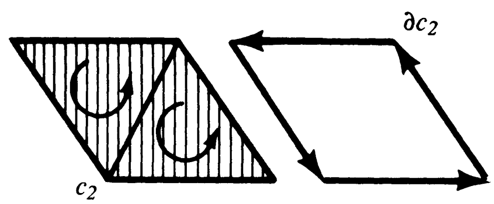

Figure 153 Boundary of a chain

PROBLEM 10. Show that the boundary of the boundary of any chain is zero: $\partial\partial c_k = 0$.

Hint. By the linearity of ∂ it is enough to show that $\partial\partial D = 0$ for a convex polyhedron D. It remains to verify that every $(k-2)$-dimensional face of D appears in $\partial\partial D$ twice, with opposite signs. It is enough to prove this for $k = 2$ (planar cross-sections).

G *The integral of a form over a chain*

Let ω^k be a k-form on M, and c_k a k-chain on M, $c_k = \sum m_i\sigma_i$. The *integral of the form ω^k over the chain c_k* is the sum of the integrals on the cells, counting multiplicities:

$$\int_{c_k} \omega^k = \sum m_i \int_{\sigma_i} \omega^k.$$

PROBLEM 11. Show that the integral depends linearly on the form:

$$\int_{c_k} \omega_1^k + \omega_2^k = \int_{c_k} \omega_1^k + \int_{c_k} \omega_2^k.$$

PROBLEM 12. Show that integration of a fixed form ω^k on chains c_k defines a homomorphism from the group of chains to the line.

[57] We are taking $k > 1$ here. One-dimensional chains are included in the general scheme if we make the following definitions: a zero-dimensional chain consists of a collection of points with multiplicities; the boundary of an oriented interval $\overrightarrow{AB}$ is $B - A$ (the point B with multiplicity 1 and A with multiplicity -1); the boundary of a point is empty.

EXAMPLE 1. Let M be the plane $\{(p, q)\}$, ω^1 the form pdq, and c_1 the chain consisting of one cell σ with multiplicity 1:

$$[0 \leq t \leq 2\pi] \xrightarrow{f} (p = \cos t, q = \sin t).$$

Then $\int_{c_1} pdq = \pi$. In general, if a chain c_1 represents the boundary of a region G (Figure 154), then $\int_{c_1} pdq$ is equal to the area of G with sign $+$ or $-$ depending on whether the pair of vectors (outward normal, oriented boundary vector) has the same or opposite orientation as the pair (p axis, q axis).

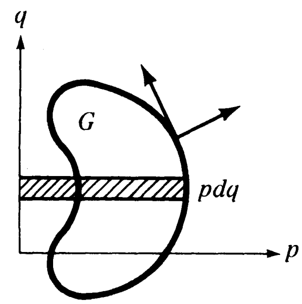

Figure 154 The integral of the form $p\,dq$ over the boundary of a region is equal to the area of the region.

EXAMPLE 2. Let M be the oriented three-dimensional euclidean space $\mathbb{R}^3$. Then every 1-form on M corresponds to some vector field $\mathbf{A}$ ($\omega^1 = \omega_{\mathbf{A}}^1$), where

$$\omega_{\mathbf{A}}^1(\boldsymbol{\xi}) = (\mathbf{A}, \boldsymbol{\xi}).$$

The integral of $\omega_{\mathbf{A}}^1$ on a chain c_1 representing a curve l is called the *circulation of the field* $\mathbf{A}$ *over the curve* l:

$$\int_{c_1} \omega_{\mathbf{A}}^1 = \int_l (\mathbf{A}, dl).$$

Every 2-form on M also corresponds to some field $\mathbf{A}$ ($\omega^2 = \omega_{\mathbf{A}}^2$, where $\omega_{\mathbf{A}}^2(\boldsymbol{\xi}, \boldsymbol{\eta}) = (\mathbf{A}, \boldsymbol{\xi}, \boldsymbol{\eta})$).
The integral of the form $\omega_{\mathbf{A}}^2$ on a chain c_2 representing an oriented surface S is called *the flux of the field* $\mathbf{A}$ *through the surface* S:

$$\int_{c_2} \omega_{\mathbf{A}}^2 = \int_S (\mathbf{A}, d\mathbf{n}).$$

PROBLEM 13. Find the flux of the field $\mathbf{A} = (1/R^2)\mathbf{e}_R$ over the surface of the sphere $x^2 + y^2 + z^2 = 1$, oriented by the vectors $\mathbf{e}_x, \mathbf{e}_y$ at the point $z = 1$. Find the flux of the same field over the surface of the ellipsoid $(x^2/a^2) + (y^2/b^2) + z^2 = 1$ oriented the same way.
Hint. Cf. Section 36H.

PROBLEM 14. Suppose that, in the $2n$-dimensional space $\mathbb{R}^n = \{(p_1, \ldots, p_n; q_1, \ldots, q_n)\}$, we are given a 2-chain c_2 representing a two-dimensional oriented surface S with boundary l. Find

$$\int_{c_2} dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n \text{ and } \int_l p_1 dq_1 + \cdots + p_n dq_n.$$

ANSWER. The sum of the oriented areas of the projection of S on the two-dimensional coordinate planes p_i, q_i.

36 Exterior differentiation

We define here exterior differentiation of k-forms and prove Stokes' theorem: the integral of the derivative of a form over a chain is equal to the integral of the form itself over the boundary of the chain.

A *Example: the divergence of a vector field*

The exterior derivative of a k-form ω on a manifold M is a $(k + 1)$-form $d\omega$ on the same manifold. Going from a form to its exterior derivative is analogous to forming the differential of a function or the divergence of a vector field. We recall the definition of divergence.

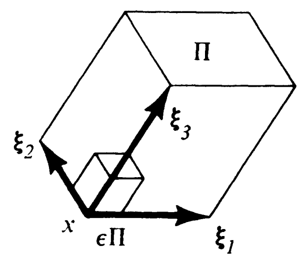

Figure 155 Definition of divergence of a vector field

Let $\mathbf{A}$ be a vector field on the oriented euclidean three-space $\mathbb{R}^3$, and let S be the boundary of a parallelepiped Π with edges ξ_1, ξ_2, and ξ_3 at the vertex x (Figure 155). Consider the ("outward") flux of the field $\mathbf{A}$ through the surface S:

$$F(\Pi) = \int_S (\mathbf{A}, d\mathbf{n}).$$

If the parallelepiped Π is very small, the flux F is approximately proportional to the product of the volume of the parallelepiped, $V = (\xi_1, \xi_2, \xi_3)$, and the "source density" at the point x. This is the limit

$$\lim_{\varepsilon \to 0} \frac{F(\varepsilon\Pi)}{\varepsilon^3 V}$$

where $\varepsilon\Pi$ is the parallelepiped with edges $\varepsilon\xi_1$, $\varepsilon\xi_2$, $\varepsilon\xi_3$. This limit does not depend on the choice of the parallelepiped Π but only on the point x, and is called the *divergence*, div $\mathbf{A}$, of the field $\mathbf{A}$ at x.

To go to higher-dimensional cases, we note that the "flux of $\mathbf{A}$ through a surface element" is the 2-form which we called $\omega_{\mathbf{A}}^2$. The divergence, then, is the density in the expression for the 3-form

$$\omega^3 = \operatorname{div} \mathbf{A}\, dx \wedge dy \wedge dz,$$

$$\omega^3(\xi_1, \xi_2, \xi_3) = \operatorname{div} \mathbf{A} \cdot V(\xi_1, \xi_2, \xi_3),$$

characterizing the "sources in an elementary parallelepiped."

The exterior derivative $d\omega^k$ of a k-form ω^k on an n-dimensional manifold M may be defined as the principal multilinear part of the integral of ω^k over the boundaries of $(k + 1)$-dimensional parallelepipeds.

B *Definition of the exterior derivative*

We define the value of the form $d\omega$ on $k + 1$ vectors $\xi_1, \ldots, \xi_{k+1}$ tangent to M at $\mathbf{x}$. To do this, we choose some coordinate system in a neighborhood of $\mathbf{x}$ on M, i.e., a differentiable map f of a neighborhood of the point 0 in euclidean space $\mathbb{R}^n$ to a neighborhood of $\mathbf{x}$ in M (Figure 156).

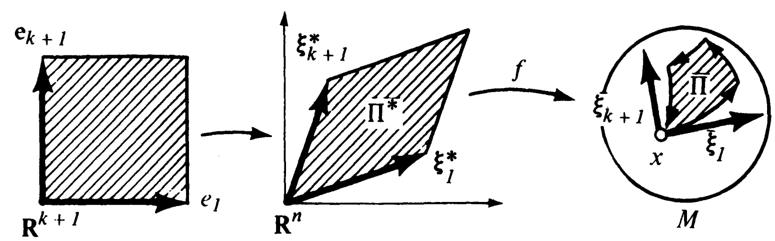

Figure 156 The curvilinear parallelepiped Π.

The pre-images of the vectors $\xi_1, \ldots, \xi_{k+1} \in TM_{\mathbf{x}}$ under the differential of f lie in the tangent space to $\mathbb{R}^n$ at 0. This tangent space can be naturally identified with $\mathbb{R}^n$, so we may consider the pre-images to be vectors

$$\xi_1^*, \ldots, \xi_{k+1}^* \in \mathbb{R}^n.$$

We take the parallelepiped Π^* in $\mathbb{R}^n$ spanned by these vectors (strictly speaking, we must look at the standard oriented cube in $\mathbb{R}^{k+1}$ and its linear map onto Π^*, taking the edges $\mathbf{e}_1, \ldots, \mathbf{e}_{k+1}$ to $\xi_1^*, \ldots, \xi_{k+1}^*$, as a $(k + 1)$-dimensional cell in $\mathbb{R}^n$). The map f takes the parallelepiped Π to a $(k + 1)$-dimensional cell on M (a "curvilinear parallelepiped"). The boundary of the cell Π is a k-chain, $\partial\Pi$. Consider the integral of the form ω^k on the boundary $\partial\Pi$ of Π:

$$F(\xi_1, \ldots, \xi_{k+1}) = \int_{\partial\Pi} \omega^k.$$

Example. We will call a smooth function $\varphi: M \to R$ a 0-form on M. The integral of the 0-form φ on the 0-chain $c_0 = \sum m_i A_i$ (where the m_i are integers and the A_i points of M) is

$$\int_{c_0} \varphi = \sum m_i \varphi(A_i).$$

Then the definition above gives the "increment" $F(\xi_1) = \varphi(x_1) - \varphi(x)$ (Figure 157) of the function φ, and the principal linear part of $F(\xi_1)$ at 0 is simply the differential of φ.

Problem 1. Show that the function $F(\xi_1, \ldots, \xi_{k+1})$ is skew-symmetric with respect to ξ.

It turns out that the principal $(k + 1)$-linear part of the "increment" $F(\xi_1, \ldots, \xi_{k+1})$ is an exterior $(k + 1)$-form on the tangent space $TM_{\mathbf{x}}$ to M

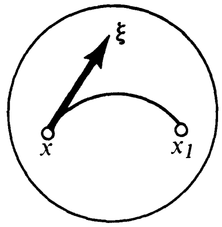

Figure 157 The integral over the boundary of a one-dimensional parallelepiped is the change in the function.

at $\mathbf{x}$. This form does not depend on the coordinate system that was used to define the curvilinear parallelepiped Π. It is called the *exterior derivative*, or *differential*, of the form ω^k (at the point $\mathbf{x}$) and is denoted by $d\omega^k$.

C *A theorem on exterior derivatives*

Theorem. *There is a unique $(k+1)$-form Ω on $TM_{\mathbf{x}}$ which is the principal $(k+1)$-linear part at 0 of the integral over the boundary of a curvilinear parallelepiped, $F(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{k+1})$; i.e.,*

(1) $$F(\varepsilon\boldsymbol{\xi}_1, \ldots, \varepsilon\boldsymbol{\xi}_{k+1}) = \varepsilon^{k+1}\Omega(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{k+1}) + o(\varepsilon^{k+1}) \qquad (\varepsilon \to 0).$$

The form Ω does not depend on the choice of coordinates involved in the definition of F. If, in the local coordinate system $x_1, \ldots, x_n$ on M, the form ω^k is written as

$$\omega^k = \sum a_{i_1, \ldots, i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

then Ω is written as

(2) $$\Omega = d\omega^k = \sum da_{i_1, \ldots, i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

We will carry out the proof of this theorem for the case of a form $\omega^1 = a(x_1, x_2)dx_1$ on the x_1, x_2 plane. The proof in the general case is entirely analogous, but the calculations are somewhat longer.

We calculate $F(\boldsymbol{\xi}, \boldsymbol{\eta})$, i.e., the integral of ω^1 on the boundary of the parallelogram Π with sides $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ and vertex at 0 (Figure 158). The chain $\partial\Pi$ is

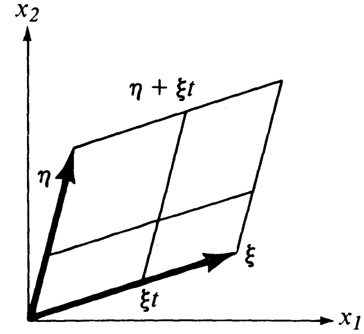

Figure 158 Theorem on exterior derivatives

given by the mappings of the interval $0 \leq t \leq 1$ to the plane $t \to \boldsymbol{\xi}t$, $t \to \boldsymbol{\xi} + \boldsymbol{\eta}t$, $t \to \boldsymbol{\eta}t$, and $t \to \boldsymbol{\eta} + \boldsymbol{\xi}t$ with multiplicities $1, 1, -1$, and -1. Therefore,

$$\int_{\partial\Pi} \omega^1 = \int_0^1 [a(\boldsymbol{\xi}t) - a(\boldsymbol{\xi}t + \boldsymbol{\eta})]\xi_1 - [a(\boldsymbol{\eta}t) - a(\boldsymbol{\eta}t + \boldsymbol{\xi})]\eta_1 \, dt$$

where $\xi_1 = dx_1(\boldsymbol{\xi})$, $\eta_1 = dx_1(\boldsymbol{\eta})$, $\xi_2 = dx_2(\boldsymbol{\xi})$, and $\eta_2 = dx_2(\boldsymbol{\eta})$ are the components of the vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. But

$$a(\boldsymbol{\xi}t + \boldsymbol{\eta}) - a(\boldsymbol{\xi}t) = \frac{\partial a}{\partial x_i}\eta_1 + \frac{\partial a}{\partial x_2}\eta_2 + O(\boldsymbol{\xi}^2, \boldsymbol{\eta}^2)$$

(the derivatives are taken at $x_1 = x_2 = 0$). In the same way

$$a(\boldsymbol{\eta}t + \boldsymbol{\xi}) - a(\boldsymbol{\eta}t) = \frac{\partial a}{\partial x_1}\xi_1 + \frac{\partial a}{\partial x_2}\xi_2 + O(\boldsymbol{\xi}^2, \boldsymbol{\eta}^2).$$

By using these expressions in the integral, we find that

$$F(\boldsymbol{\xi}, \boldsymbol{\eta}) = \int_{\partial\Pi} \omega^1 = \frac{\partial a}{\partial x_2}(\xi_2\eta_1 - \xi_1\eta_2) + o(\boldsymbol{\xi}^2, \boldsymbol{\eta}^2).$$

The principal bilinear part of F, as promised in (1), turns out to be the value of the exterior 2-form

$$\Omega = \frac{\partial a}{\partial x_2} dx_2 \wedge dx_1$$

on the pair of vectors $\boldsymbol{\xi}$, $\boldsymbol{\eta}$. Thus the form obtained is given by formula (2), since

$$da \wedge dx_1 = \frac{\partial a}{\partial x_1} dx_1 \wedge dx_1 + \frac{\partial a}{\partial x_2} dx_2 \wedge dx_1 = \frac{\partial a}{\partial x_2} dx_2 \wedge dx_1.$$

Finally, if the coordinate system x_1, x_2 is changed to another (Figure 159), the parallelogram Π is changed to a nearby curvilinear parallelogram Π', so that the difference in the values of the integrals, $\int_{\partial\Pi} \omega^1 - \int_{\partial\Pi'} \omega^1$ will be small of more than second order (prove it!). □

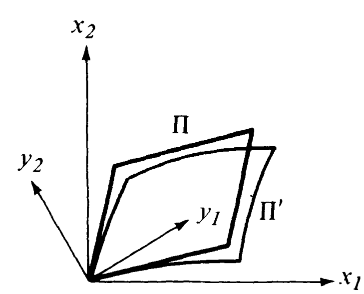

Figure 159 Independence of the exterior derivative from the coordinate system

Problem 2. Carry out the proof of the theorem in the general case.

Problem 3. Prove the formulas for differentiating a sum and a product:

$$d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2.$$

and

$$d(\omega^k \wedge \omega^l) = d\omega^k \wedge \omega^l + (-1)^k \omega^k \wedge d\omega^l,$$

Problem 4. Show that the differential of a differential is equal to zero: $dd = 0$.

Problem 5. Let $f\colon M \to N$ be a smooth map and ω a k-form on N. Show that $f^*(d\omega) = d(f^*\omega)$.

D *Stokes' formula*

One of the most important corollaries of the theorem on exterior derivatives is the Newton-Leibniz-Gauss-Green-Ostrogradskii-Stokes-Poincaré formula:

$$(3) \qquad \int_{\partial c} \omega = \int_c d\omega,$$

where c is any $(k + 1)$-chain on a manifold M and ω is any k-form on M.

To prove this formula it is sufficient to prove it for the case when the chain consists of one cell σ. We assume first that this cell σ is given by an oriented parallelepiped $\Pi \subset \mathbb{R}^{k+1}$ (Figure 160).

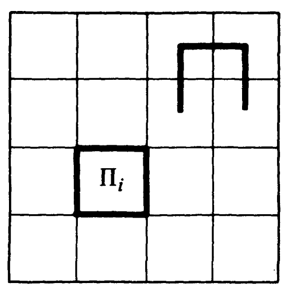

Figure 160 Proof of Stokes' formula for a parallelepiped

We partition Π into N^{k+1} small equal parallelepipeds Π_i similar to Π. Then, clearly,

$$\int_{\partial\Pi} \omega = \sum_{i=1}^{N^{k+1}} F_i, \quad \text{where } F_i = \int_{\partial\Pi_i} \omega.$$

By formula (1) we have

$$F_i = d\omega(\xi_1^i, \dots, \xi_{k+1}^i) + o(N^{-(k+1)}),$$

where $\xi_1^i, \dots, \xi_{k+1}^i$ are the edges of Π_i. But $\sum_{i=1}^{N^{k+1}} d\omega(\xi_1^i, \dots, \xi_{k+1}^i)$ is a Riemann sum for $\int_\Pi d\omega$. It is easy to verify that $o(N^{-(k+1)})$ is uniform, so

$$\lim_{N\to\infty} \sum_{i=1}^{N^{k+1}} F_i = \lim_{N\to\infty} \sum_{i=1}^{N^{k+1}} d\omega(\xi_1^i, \dots, \xi_{k+1}^i) = \int_\Pi d\omega.$$

Finally, we obtain

$$\int_{\partial \Pi} \omega = \sum F_i = \lim_{N \to \infty} \sum F_i = \int_{\Pi} d\omega.$$

Formula (3) follows automatically from this for any chain whose polyhedra are parallelepipeds.

To prove formula (3) for any convex polyhedron D, it is enough to prove it for a simplex,[58] since D can always be partitioned into simplices (Figure 161):

$$D = \sum D_i \qquad \partial D = \sum \partial D_i.$$

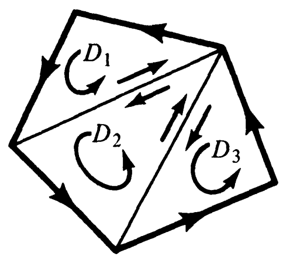

Figure 161 Division of a convex polyhedron into simplices

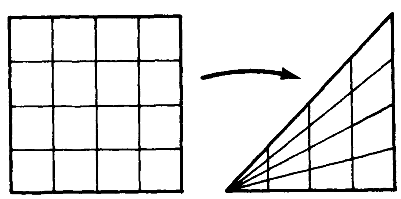

Figure 162 Proof of Stokes' formula for a simplex

We will prove formula (3) for a simplex. Notice that a k-dimensional oriented cube can be mapped onto a k-dimensional simplex so that:

1. The interior of the cube goes diffeomorphically, with its orientation preserved, onto the interior of the simplex;
2. The interiors of some $(k-1)$-dimensional faces of the cube go diffeomorphically, with their orientations preserved, onto the interiors of the faces of the simplex; the images of the remaining $(k-1)$-dimensional faces of the cube lie in the $(k-2)$-dimensional faces of the simplex.

For example, for $k = 2$ such a map of the cube $0 \leq x_1, x_2 \leq 1$ onto the triangle is given by the formula $y_1 = x_1$, $y_2 = x_1 x_2$ (Figure 162). Then,

[58] A two-dimensional simplex is a triangle, a three-dimensional simplex is a tetrahedron, a k-dimensional simplex is the convex hull of $k+1$ points in $\mathbb{R}^n$ which do not lie in any $k-1$-dimensional plane.

EXAMPLE: $\{x \in \mathbb{R}^k : x_i \geq 0 \text{ and } \sum_{i=1}^k x_i \leq 1\}$.

formula (3) for the simplex follows from formula (3) for the cube and the change of variables theorem (cf. Section 35C).

EXAMPLE 1. Consider the 1-form

$$\omega^1 = p_1\,dq_1 + \cdots + p_n\,dq_n = \mathbf{p}\,d\mathbf{q}$$

on $\mathbb{R}^{2n}$ with coordinates $p_1, \ldots, p_n, q_1, \ldots, q_n$. Then $d\omega^1 = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n = d\mathbf{p} \wedge d\mathbf{q}$, so

$$\iint_{c_2} d\mathbf{p} \wedge d\mathbf{q} = \int_{\partial c_2} \mathbf{p}\,d\mathbf{q}.$$

In particular, if c_2 is a closed surface ($\partial c_2 = 0$), then $\iint_{c_2} d\mathbf{p} \wedge d\mathbf{q} = 0$.

E *Example 2—Vector analysis*

In a three-dimensional oriented riemannian space M, every vector field $\mathbf{A}$ corresponds to a 1-form $\omega_{\mathbf{A}}^1$ and a 2-form $\omega_{\mathbf{A}}^2$. Therefore, exterior differentiation can be considered as an operation on vectors.

Exterior differentiation of 0-forms (functions), 1-forms, and 2-forms correspond to the operations of gradient, curl, and divergence defined by the relations

$$df = \omega^1_{\mathbf{grad}\,f} \qquad d\omega^1_{\mathbf{A}} = \omega^2_{\mathbf{curl}\,\mathbf{A}} \qquad d\omega^2_{\mathbf{A}} = (\operatorname{div}\mathbf{A})\omega^3$$

(the form ω^3 is the volume element on M). Thus, it follows from (3) that

$$f(y) - f(x) = \int_l \mathbf{grad}\,f\,d\mathbf{l} \quad \text{if } \partial l = y - x$$

$$\int_l \mathbf{A}\,d\mathbf{l} = \iint_S \mathbf{curl}\,\mathbf{A} \cdot d\mathbf{n} \quad \text{if } \partial S = l$$

$$\iint_S \mathbf{A}\,d\mathbf{n} = \iiint_D (\operatorname{div}\mathbf{A})\omega^3 \quad \text{if } \partial D = S.$$

PROBLEM 5. Show that

$$\operatorname{div}[\mathbf{A}, \mathbf{B}] = (\mathbf{curl}\,\mathbf{A}, \mathbf{B}) - (\mathbf{curl}\,\mathbf{B}, \mathbf{A}),$$

$$\mathbf{curl}\,a\mathbf{A} = [\mathbf{grad}\,a, \mathbf{A}] + a\,\mathbf{curl}\,\mathbf{A},$$

$$\operatorname{div} a\mathbf{A} = (\mathbf{grad}\,a, \mathbf{A}) + a \operatorname{div}\mathbf{A}.$$

Hint. By the formula for differentiating the product of forms,

$$d(\omega^2_{[\mathbf{A},\mathbf{B}]}) = d(\omega^1_{\mathbf{A}} \wedge \omega^1_{\mathbf{B}}) = d\omega^1_{\mathbf{A}} \wedge \omega^1_{\mathbf{B}} - \omega^1_{\mathbf{A}} \wedge d\omega^1_{\mathbf{B}}.$$

PROBLEM 6. Show that $\mathbf{curl}\,\mathbf{grad} = \operatorname{div}\mathbf{curl} = 0$.
Hint. $dd = 0$.

F *Appendix 1: Vector operations in triply orthogonal systems*

Let x_1, x_2, x_3 be a triply orthogonal coordinate system on M, $ds^2 = E_1\,dx_1^2 + E_2\,dx_2^2 + E_3\,dx_3^2$ and $\mathbf{e}_i$ the coordinate unit vectors (cf. Section 34F).

Problem 7. Given the components of a vector field $\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3$, find the components of its curl.

Solution. According to Section 34F

$$\omega_{\mathbf{A}}^1 = A_1\sqrt{E_1}\,dx_1 + A_2\sqrt{E_2}\,dx_2 + A_3\sqrt{E_3}\,dx_3.$$

Therefore,

$$d\omega_{\mathbf{A}}^1 = \left(\frac{\partial A_3\sqrt{E_3}}{\partial x_2} - \frac{\partial A_2\sqrt{E_2}}{\partial x_3}\right)dx_2 \wedge dx_3 + \cdots = \omega_{\mathbf{curl\,A}}^2.$$

According to Section 34F, we have

$$\mathbf{curl\,A} = \frac{1}{\sqrt{E_2E_3}}\left(\frac{\partial A_3\sqrt{E_3}}{\partial x_2} - \frac{\partial A_2\sqrt{E_2}}{\partial x_3}\right)\mathbf{e}_1 + \cdots = \frac{1}{\sqrt{E_1E_2E_3}}\begin{vmatrix} \sqrt{E_1}\,\mathbf{e}_1 & \sqrt{E_2}\,\mathbf{e}_2 & \sqrt{E_3}\,\mathbf{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ A_1\sqrt{E_1} & A_2\sqrt{E_2} & A_3\sqrt{E_3} \end{vmatrix}.$$

In particular, in cartesian, cylindrical, and spherical coordinates on $\mathbb{R}^3$,

$$\begin{aligned}\mathbf{curl\,A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{e}_z \\ &= \frac{1}{r}\left(\frac{\partial A_z}{\partial \varphi} - \frac{\partial r A_\varphi}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\mathbf{e}_\varphi + \frac{1}{r}\left(\frac{\partial r A_\varphi}{\partial r} - \frac{\partial A_r}{\partial \varphi}\right)\mathbf{e}_z \\ &= \frac{1}{R\cos\theta}\left(\frac{\partial A_\theta}{\partial \varphi} - \frac{\partial A_\varphi \cos\theta}{\partial \theta}\right)\mathbf{e}_R + \frac{1}{R}\left(\frac{\partial A_R}{\partial \theta} - \frac{\partial R A_\theta}{\partial R}\right)\mathbf{e}_\varphi + \frac{1}{R}\left(\frac{\partial R A_\varphi}{\partial R} - \frac{1}{\cos\theta}\frac{\partial A_R}{\partial \varphi}\right)\mathbf{e}_\theta.\end{aligned}$$

Problem 9. Find the divergence of the field $\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3$.

Solution. $\omega_{\mathbf{A}}^2 = A_1\sqrt{E_2E_3}\,dx_2 \wedge dx_3 + \cdots$. Therefore,

$$d\omega_{\mathbf{A}}^2 = \frac{\partial}{\partial x_1}(A_1\sqrt{E_2E_3})\,dx_1 \wedge dx_2 \wedge dx_3 + \cdots.$$

By the definition of divergence,

$$d\omega_{\mathbf{A}}^2 = \operatorname{div}\mathbf{A}\sqrt{E_1E_2E_3}\,dx_1 \wedge dx_2 \wedge dx_3.$$

This means

$$\operatorname{div}\mathbf{A} = \frac{1}{\sqrt{E_1E_2E_3}}\left(\frac{\partial}{\partial x_1}A_1\sqrt{E_2E_3} + \frac{\partial}{\partial x_2}A_2\sqrt{E_3E_1} + \frac{\partial}{\partial x_3}A_3\sqrt{E_1E_2}\right).$$

In particular, in cartesian, cylindrical, and spherical coordinates on $\mathbb{R}^3$:

$$\operatorname{div} \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{1}{r}\left(\frac{\partial r A_r}{\partial r} + \frac{\partial A_\varphi}{\partial \varphi}\right) + \frac{\partial A_z}{\partial z}$$
$$= \frac{1}{R^2 \cos\theta}\left(\frac{\partial R^2 \cos\theta A_R}{\partial R} + \frac{\partial R A_\varphi}{\partial \varphi} + \frac{\partial R \cos\theta A_\theta}{\partial \theta}\right).$$

PROBLEM 10. The Laplace operator on M is the operator $\Delta = \operatorname{div} \operatorname{grad}$. Find its expression in the coordinates x_i.

ANSWER.

$$\Delta f = \frac{1}{\sqrt{E_1 E_2 E_3}}\left[\frac{\partial}{\partial x_1}\left(\sqrt{\frac{E_2 E_3}{E_1}}\frac{\partial f}{\partial x_1}\right) + \cdots\right].$$

In particular, on $\mathbb{R}^3$

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$
$$= \frac{1}{R^2 \cos\theta}\left[\frac{\partial}{\partial R}\left(R^2 \cos\theta \frac{\partial f}{\partial R}\right) + \frac{\partial}{\partial \varphi}\left(\frac{1}{\cos\theta}\frac{\partial f}{\partial \varphi}\right) + \frac{\partial}{\partial \theta}\left(\cos\theta \frac{\partial f}{\partial \theta}\right)\right].$$

G *Appendix 2: Closed forms and cycles*

The flux of an incompressible fluid (without sources) across the boundary of a region D is equal to zero. We will formulate a higher-dimensional analogue to this obvious assertion. The higher-dimensional analogue of an incompressible fluid is called a *closed form*. The field $\mathbf{A}$ has no sources if $\operatorname{div} \mathbf{A} = 0$.

Definition. A differential form ω on a manifold M is *closed* if its exterior derivative is zero: $d\omega = 0$.

In particular, the 2-form $\omega_{\mathbf{A}}^2$ corresponding to a field $\mathbf{A}$ without sources is closed. Also, we have, by Stokes' formula (3):

Theorem. *The integral of a closed form ω^k over the boundary of any $(k+1)$-dimensional chain c_{k+1} is equal to zero:*

$$\int_{\partial c_{k+1}} \omega^k = 0 \quad \textit{if } d\omega^k = 0.$$

PROBLEM 11. Show that the differential of a form is always closed.

On the other hand, there are closed forms which are not differentials. For example, take for M the three-dimensional euclidean space $\mathbb{R}^3$ without 0: $M = \mathbb{R}^3 - 0$, with the 2-form being the flux of the field $\mathbf{A} = (1/R^2)\mathbf{e}_R$ (Figure 163). It is easy to convince oneself that $\operatorname{div} \mathbf{A} = 0$, so that our 2-form

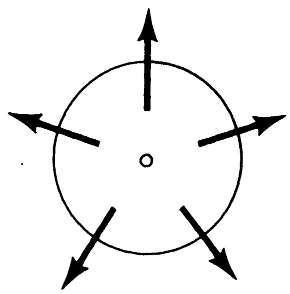

Figure 163 The field **A**

$\omega_{\mathbf{A}}^2$ is closed. At the same time, the flux over any sphere with center 0 is equal to 4π. We will show that the integral of the differential of a form over the sphere must be zero.

Definition. A *cycle* on a manifold M is a chain whose boundary is equal to zero.

The oriented surface of our sphere can be considered to be a cycle. It immediately follows from Stokes' formula (3) that

Theorem. *The integral of a differential over any cycle is equal to zero:*

$$\int_{c_{k+1}} d\omega^k = 0 \quad \textit{if } \partial c_{k+1} = 0.$$

Thus, our 2-form $\omega_{\mathbf{A}}^2$ is not the differential of any 1-form.

The existence of closed forms on M which are not differentials is related to the topological properties of M. One can show that every closed k-form on a vector space is the differential of some $(k + 1)$-form (Poincaré's lemma).

PROBLEM 12. Prove Poincaré's lemma for 1-forms.
Hint. Consider $\int_{x_0}^{x_1} \omega^1 = \varphi(x_1)$.

PROBLEM 13. Show that in a vector space the integral of a closed form over any cycle is zero.
Hint. Construct a $(k + 1)$-chain whose boundary is the given cycle (Figure 164).

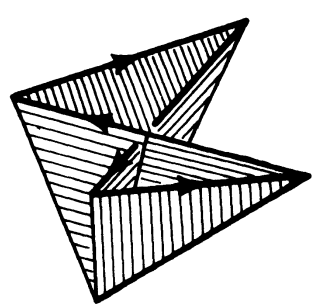

Figure 164 Cone over a cycle

Namely, for any chain c consider the "cone over c with vertex 0." If we denote the operation of constructing a cone by p, then

$$\partial \circ p + p \circ \partial = 1 \qquad \text{(the identity map).}$$

Therefore, if the chain c is closed, $\partial(pc) = c$.

Problem. Show that every closed form on a vector space is an exterior derivative.

Hint. Use the cone construction. Let ω^k be a differential k-form on $\mathbb{R}^n$. We define a $(k-1)$-form (the "co-cone over ω") $p\omega^k$ in the following way: for any chain c_{k-1}

$$\int_{c_{k-1}} p\omega^k = \int_{pc_k} \omega^k.$$

It is easy to see that the $(k-1)$-form $p\omega^k$ exists and is unique; its value on the vectors $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_k$, tangent to $\mathbb{R}^n$ at $\mathbf{x}$, is equal to

$$(p\omega)_{\mathbf{x}}(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_k) = \int_0^1 \omega_{t\mathbf{x}}(\mathbf{x}, t\boldsymbol{\xi}_1, \ldots, t\boldsymbol{\xi}_{k-1})dt.$$

It is easy to see that

$$d \circ p + p \circ d = 1 \qquad \text{(the identity map).}$$

Therefore, if the form ω^k is closed, $d(p\omega^k) = \omega^k$.

Problem. Let $\mathbf{X}$ be a vector field on M and ω a differential k-form. We define a differential $(k-1)$-form $i_{\mathbf{X}}\omega$ (the *interior derivative* of ω by $\mathbf{X}$) by the relation

$$(i_{\mathbf{X}}\omega)(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{k-1}) = \omega(\mathbf{X}, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{k-1}).$$

Prove the *homotopy formula*

$$i_{\mathbf{X}}d + di_{\mathbf{X}} = L_{\mathbf{X}},$$

where $L_{\mathbf{X}}$ is the differentiation operator in the direction of the field $\mathbf{X}$.

[The action of $L_{\mathbf{X}}$ on a form is defined, using the phase flow $\{g^t\}$ of the field $\mathbf{X}$, by the relation

$$(L_{\mathbf{X}}\omega)(\boldsymbol{\xi}) = \left.\frac{d}{dt}\right|_{t=0} \omega(g^t_* \boldsymbol{\xi}).$$

L_X is called the *Lie derivative* or *fisherman's derivative*: the flow carries all possible differential-geometric objects past the fisherman, and the fisherman sits there and differentiates them.]

Hint. We denote by H the "homotopy operator" associating to a k-chain $\gamma: \sigma \to M$ the $(k+1)$-chain $H\gamma: (I \times \sigma) \to M$ according to the formula $(H\gamma)(t, x) = g^t\gamma(x)$ (where $I = [0, 1]$). Then

$$g^1\gamma - \gamma = \partial(H\gamma) + H(\partial\gamma).$$

Problem. Prove the formula for differentiating a vector product on three-dimensional euclidean space (or on a riemannian manifold):

$$\mathbf{curl}[\mathbf{a}, \mathbf{b}] = \{\mathbf{a}, \mathbf{b}\} + \mathbf{a} \operatorname{div} \mathbf{b} - \mathbf{b} \operatorname{div} \mathbf{a}$$

(where $\{\mathbf{a}, \mathbf{b}\} = L_{\mathbf{a}}\mathbf{b}$ is the Poisson bracket of the vector fields, cf. Section 39).

Hint. If τ is the volume element, then

$$i_{\mathbf{curl}[\mathbf{a},\mathbf{b}]}\tau = di_{\mathbf{a}}i_{\mathbf{b}}\tau \qquad \operatorname{div} \mathbf{a} = di_{\mathbf{a}}\tau \quad \text{and} \quad \{\mathbf{a}, \mathbf{b}\} = L_{\mathbf{a}}\mathbf{b};$$

by using these relations and the fact that $d\tau = 0$, it is easy to derive the formula for $\mathbf{curl}[\mathbf{a}, \mathbf{b}]$ from the homotopy formula.

H *Appendix 3: Cohomology and homology*

The set of all k-forms on M forms a vector space, the closed k-forms a subspace and the differentials of $(k + 1)$-forms a subspace of the subspace of closed forms. The quotient space

$$\frac{\text{(closed forms)}}{\text{(differentials)}} = H^k(M, \mathbb{R})$$

is called the *k-th cohomology group* of the manifold M. An element of this group is a class of closed forms differing from one another only by a differential.

PROBLEM 14. Show that for the circle S^1 we have $H^1(S^1, \mathbb{R}) = \mathbb{R}$.

The dimension of the space $H^k(M, \mathbb{R})$ is called the *k-th Betti number* of M.

PROBLEM 15. Find the first Betti number of the torus $T^2 = S^1 \times S^1$.

The flux of an incompressible fluid (without sources) over the surfaces of two concentric spheres is the same. In general, when integrating a closed form

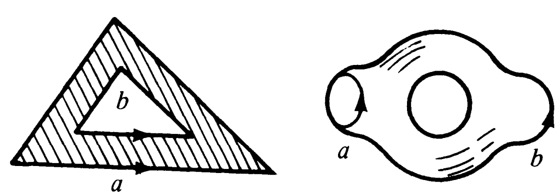

Figure 165 Homologous cycles

over a k-dimensional cycle, we can replace the cycle with another one provided that their difference is the boundary of a $(k + 1)$-chain (Figure 165):

$$\int_a \omega^k = \int_b \omega^k,$$

if $a - b = \partial c_{k+1}$ and $d\omega^k = 0$.

Poincaré called two such cycles a and b *homologous*.

With a suitable definition[59] of the group of chains on a manifold M and its

[59] For this our group $\{c_k\}$ must be made smaller by identifying pieces which differ only by the choice of parametrization f or the choice of polyhedron D. In particular, we may assume that D is always one and the same simplex or cube. Furthermore, we must take every degenerate k-cell (D, f, Or) to be zero, i.e., $(D, f, \text{Or}) = 0$ if $f = f_2 \cdot f_1$, where $f_1: D \to D'$ and D' has dimension smaller than k.

subgroups of cycles and boundaries (i.e., cycles homologous to zero), the quotient group

$$\frac{\text{(cycles)}}{\text{(boundaries)}} = H_k(M)$$

is called the *k-th homology group* of M.

An element of this group is a class of cycles homologous to one another.

The rank of this group is also equal to the *k-th Betti number* of M ("De-Rham's Theorem").