DETERMINANTS

The notion of a determinant appeared at the end of 17th century in works of **Leibniz** (1646–1716) and a Japanese mathematician, **Seki Kova**, also known as **Takakazu** (1642–1708). Leibniz did not publish the results of his studies related with determinants. The best known is his letter to **l'Hospital** (1693) in which Leibniz writes down the determinant condition of compatibility for a system of three linear equations in two unknowns. Leibniz particularly emphasized the usefulness of two indices when expressing the coefficients of the equations. In modern terms he actually wrote about the indices i, j in the expression $x_i = \sum_j a_{ij} y_j$.

Seki arrived at the notion of a determinant while solving the problem of finding common roots of algebraic equations.

In Europe, the search for common roots of algebraic equations soon also became the main trend associated with determinants. Newton, Bezout, and Euler studied this problem.

Seki did not have the general notion of the derivative at his disposal, but he actually got an algebraic expression equivalent to the derivative of a polynomial. He searched for multiple roots of a polynomial $f(x)$ as common roots of $f(x)$ and $f'(x)$. To find common roots of polynomials $f(x)$ and $g(x)$ (for f and g of small degrees) Seki got determinant expressions. The main treatise by Seki was published in 1674; there applications of the method are published, rather than the method itself. He kept the main method in secret confiding only in his closest pupils.

In Europe, the first publication related to determinants, due to **Cramer**, appeared in 1750. In this work Cramer gave a determinant expression for a solution of the problem of finding the conic through 5 fixed points (this problem reduces to a system of linear equations).

The general theorems on determinants were proved only *ad hoc* when needed to solve some other problem. Therefore, the theory of determinants had been developing slowly, left behind out of proportion as compared with the general development of mathematics. A systematic presentation of the theory of determinants is mainly associated with the names of **Cauchy** (1789–1857) and **Jacobi** (1804–1851).

1. Basic properties of determinants

The *determinant* of a square matrix $A = \left\|a_{ij}\right\|_1^n$ is the alternated sum

$$\sum_\sigma (-1)^\sigma a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)},$$

where the summation is over all permutations $\sigma \in S_n$. The determinant of the matrix $A = \left\|a_{ij}\right\|_1^n$ is denoted by $\det A$ or $|a_{ij}|_1^n$. If $\det A \neq 0$, then A is called *invertible* or *nonsingular*.

The following properties are often used to compute determinants. The reader can easily verify (or recall) them.

1. Under the permutation of two rows of a matrix A its determinant changes the sign. In particular, if two rows of the matrix are identical, $\det A = 0$.

2. If A and B are square matrices, $\det \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \det A \cdot \det B$.

3. $|a_{ij}|_1^n = \sum_{j=1}^n (-1)^{i+j} a_{ij} M_{ij}$, where M_{ij} is the determinant of the matrix obtained from A by crossing out the ith row and the jth column of A (the *row (echelon) expansion* of the determinant or, more precisely, *the expansion with respect to the ith row*).

(To prove this formula one has to group the factors of a_{ij}, where $j = 1, \dots, n$, for a fixed i.)

4.

$$\begin{vmatrix} \lambda\alpha_1 + \mu\beta_1 & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \dots & \vdots \\ \lambda\alpha_n + \mu\beta_n & a_{n2} & \dots & a_{nn} \end{vmatrix} = \lambda \begin{vmatrix} \alpha_1 & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \dots & \vdots \\ \alpha_n & a_{n2} & \dots & a_{nn} \end{vmatrix} + \mu \begin{vmatrix} \beta_1 & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \dots & \vdots \\ \beta_n & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

5. $\det(AB) = \det A \det B$.
6. $\det(A^T) = \det A$.

1.1. Before we start computing determinants, let us prove *Cramer's rule*. It appeared already in the first published paper on determinants.

Theorem (Cramer's rule). *Consider a system of linear equations*

$$x_1 a_{i1} + \dots + x_n a_{in} = b_i \quad (i = 1, \dots, n),$$

i.e.,

$$x_1 A_1 + \dots + x_n A_n = B,$$

where A_j is the jth column of the matrix $A = \left\|a_{ij}\right\|_1^n$. Then

$$x_i \det(A_1, \dots, A_n) = \det\left(A_1, \dots, B, \dots, A_n\right),$$

where the column B is inserted instead of A_i.

Proof. Since for $j \neq i$ the determinant of the matrix $\det(A_1, \dots, A_j, \dots, A_n)$, a matrix with two identical columns, vanishes,

$$\begin{aligned} \det(A_1, \dots, B, \dots, A_n) &= \det\left(A_1, \dots, \textstyle\sum x_j A_j, \dots, A_n\right) \\ &= \sum x_j \det(A_1, \dots, A_j, \dots, A_n) = x_i \det(A_1, \dots, A_n). \quad \square \end{aligned}$$

If $\det(A_1, \dots, A_n) \neq 0$ the formula obtained can be used to find solutions of a system of linear equations.

1.2. One of the most often encountered determinants is the *Vandermonde determinant*, i.e., the determinant of the *Vandermonde matrix*

$$V(x_1, \dots, x_n) = \begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{vmatrix} = \prod_{i>j} (x_i - x_j).$$

To compute this determinant, let us subtract the $(k-1)$-st column multiplied by x_1 from the kth one for $k = n, n-1, \dots, 2$. The first row takes the form

$(1, 0, 0, \dots, 0)$, i.e., the computation of the Vandermonde determinant of order n reduces to a determinant of order $n-1$. Factorizing each row of the new determinant by bringing out $x_i - x_1$ we get

$$V(x_1, \dots, x_n) = \prod_{i>1}(x_i - x_1) \begin{vmatrix} 1 & x_2 & x_2^2 & \dots & x_1^{n-2} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-2} \end{vmatrix}.$$

For $n = 2$ the identity $V(x_1, x_2) = x_2 - x_1$ is obvious, hence,

$$V(x_1, \dots, x_n) = \prod_{i>j}(x_i - x_j).$$

Many of the applications of the Vandermonde determinant are occasioned by the fact that $V(x_1, \dots, x_n) = 0$ if and only if there are two equal numbers among $x_1, \dots, x_n$.

1.3. The *Cauchy determinant* $|a_{ij}|_1^n$, where $a_{ij} = (x_i + y_j)^{-1}$, is slightly more difficult to compute than the Vandermonde determinant.

Let us prove by induction that

$$|a_{ij}|_1^n = \frac{\prod_{i>j}(x_i - x_j)(y_i - y_j)}{\prod_{i,j}(x_i + y_j)}.$$

For a base of induction take $|a_{ij}|_1^1 = (x_1 + y_1)^{-1}$.

The step of induction will be performed in two stages.

First, let us subtract the last column from each of the preceding ones. We get

$$a'_{ij} = (x_i + y_j)^{-1} - (x_i + y_n)^{-1} = (y_n - y_j)(x_i + y_n)^{-1}(x_i + y_j)^{-1} \text{ for } j \neq n.$$

Let us take out of each row the factors $(x_i + y_n)^{-1}$ and take out of each column, except the last one, the factors $y_n - y_j$. As a result we get the determinant $|b_{ij}|_1^n$, where $b_{ij} = a_{ij}$ for $j \neq n$ and $b_{in} = 1$.

To compute this determinant, let us subtract the last row from each of the preceding ones. Taking out of each row, except the last one, the factors $x_n - x_i$ and out of each column, except the last one, the factors $(x_n + y_j)^{-1}$ we make it possible to pass to a Cauchy determinant of lesser size.

1.4. A matrix A of the form

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \end{pmatrix}$$

is called *Frobenius' matrix* or the *companion matrix* of the polynomial

$$p(\lambda) = \lambda^n - a_{n-1}\lambda^{n-1} - a_{n-2}\lambda^{n-2} - \dots - a_0.$$

With the help of the expansion with respect to the first row it is easy to verify by induction that

$$\det(\lambda I - A) = \lambda^n - a_{n-1}\lambda^{n-1} - a_{n-2}\lambda^{n-2} - \dots - a_0 = p(\lambda).$$

1.5. Let $b_i, i \in \mathbb{Z}$, such that $b_k = b_l$ if $k \equiv l \pmod{n}$ be given; the matrix $\left\|a_{ij}\right\|_1^n$, where $a_{ij} = b_{i-j}$, is called a *circulant matrix.*

Let $\varepsilon_1, \dots, \varepsilon_n$ be distinct nth roots of unity; let

$$f(x) = b_0 + b_1 x + \dots + b_{n-1} x^{n-1}.$$

Let us prove that the determinant of the circulant matrix $|a_{ij}|_1^n$ is equal to

$$f(\varepsilon_1) f(\varepsilon_2) \dots f(\varepsilon_n).$$

It is easy to verify that for $n = 3$ we have

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \varepsilon_1 & \varepsilon_1^2 \\ 1 & \varepsilon_2 & \varepsilon_2^2 \end{pmatrix} \begin{pmatrix} b_0 & b_2 & b_1 \\ b_1 & b_0 & b_2 \\ b_2 & b_1 & b_0 \end{pmatrix} \begin{pmatrix} f(1) & f(1) & f(1) \\ f(\varepsilon_1) & \varepsilon_1 f(\varepsilon_1) & \varepsilon_1^2 f(\varepsilon_1) \\ f(\varepsilon_2) & \varepsilon_2 f(\varepsilon_2) & \varepsilon_2^2 f(\varepsilon_2) \end{pmatrix}$$
$$= f(1) f(\varepsilon_1) f(\varepsilon_2) \begin{pmatrix} 1 & 1 & 1 \\ 1 & \varepsilon_1 & \varepsilon_1^2 \\ 1 & \varepsilon_2 & \varepsilon_2^2 \end{pmatrix}.$$

Therefore,

$$V(1, \varepsilon_1, \varepsilon_2) |a_{ij}|_1^3 = f(1) f(\varepsilon_1) f(\varepsilon_2) V(1, \varepsilon_1, \varepsilon_2).$$

Taking into account that the Vandermonde determinant $V(1, \varepsilon_1, \varepsilon_2)$ does not vanish, we have:

$$|a_{ij}|_1^3 = f(1) f(\varepsilon_1) f(\varepsilon_2).$$

The proof of the general case is similar.

1.6. A *tridiagonal* matrix is a square matrix $J = \left\|a_{ij}\right\|_1^n$, where $a_{ij} = 0$ for $|i - j| > 1$.

Let $a_i = a_{ii}$ for $i = 1, \dots, n$, let $b_i = a_{i,i+1}$ and $c_i = a_{i+1,i}$ for $i = 1, \dots, n-1$. Then the tridiagonal matrix takes the form

$$\begin{pmatrix} a_1 & b_1 & 0 & \dots & 0 & 0 & 0 \\ c_1 & a_2 & b_2 & \dots & 0 & 0 & 0 \\ 0 & c_2 & a_3 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & a_{n-2} & b_{n-2} & 0 \\ 0 & 0 & 0 & \dots & c_{n-2} & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \dots & 0 & c_{n-1} & a_n \end{pmatrix}.$$

To compute the determinant of this matrix we can make use of the following recurrent relation. Let $\Delta_0 = 1$ and $\Delta_k = |a_{ij}|_1^k$ for $k \geq 1$.

Expanding $\left\|a_{ij}\right\|_1^k$ with respect to the kth row it is easy to verify that

$$\Delta_k = a_k \Delta_{k-1} - b_{k-1} c_{k-1} \Delta_{k-2} \text{ for } k \geq 2.$$

The recurrence relation obtained indicates, in particular, that Δ_n (the determinant of J) depends not on the numbers b_i, c_j themselves but on their products of the form $b_i c_i$.

The quantity

$$(a_1 \dots a_n) = \begin{vmatrix} a_1 & 1 & 0 & \dots & 0 & 0 & 0 \\ -1 & a_2 & 1 & \dots & 0 & 0 & 0 \\ 0 & -1 & a_3 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \\ 0 & 0 & 0 & \ddots & a_{n-2} & 1 & 0 \\ 0 & 0 & 0 & \ddots & -1 & a_{n-1} & 1 \\ 0 & 0 & 0 & \dots & 0 & -1 & a_n \end{vmatrix}$$

is associated with continued fractions, namely:

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}} = \frac{(a_1 a_2 \dots a_n)}{(a_2 a_3 \dots a_n)}.$$

Let us prove this equality by induction. Clearly,

$$a_1 + \frac{1}{a_2} = \frac{(a_1 a_2)}{(a_2)}.$$

It remains to demonstrate that

$$a_1 + \cfrac{1}{\cfrac{(a_2 a_3 \dots a_n)}{(a_3 a_4 \dots a_n)}} = \frac{(a_1 a_2 \dots a_n)}{(a_2 a_3 \dots a_n)},$$

i.e., $a_1(a_2 \dots a_n) + (a_3 \dots a_n) = (a_1 a_2 \dots a_n)$. But this identity is a corollary of the above recurrence relation, since $(a_1 a_2 \dots a_n) = (a_n \dots a_2 a_1)$.

1.7. Under multiplication of a row of a square matrix by a number λ the determinant of the matrix is multiplied by λ. The determinant of the matrix does not vary when we replace one of the rows of the given matrix with its sum with any other row of the matrix. These statements allow a natural generalization to simultaneous transformations of several rows.

Consider the matrix $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where A_{11} and A_{22} are square matrices of order m and n, respectively.

Let D be a square matrix of order m and B a matrix of size $n \times m$.

Theorem. $\begin{vmatrix} DA_{11} & DA_{12} \\ A_{21} & A_{22} \end{vmatrix} = |D| \cdot |A|$ *and* $\begin{vmatrix} A_{11} & A_{12} \\ A_{21} + BA_{11} & A_{22} + BA_{12}. \end{vmatrix} = |A|$

Proof.

$$\begin{pmatrix} DA_{11} & DA_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} + BA_{11} & A_{22} + BA_{12} \end{pmatrix} = \begin{pmatrix} I & 0 \\ B & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \quad \square$$

Problems

1.1. Let $A = \left\|a_{ij}\right\|_1^n$ be skew-symmetric, i.e., $a_{ij} = -a_{ji}$, and let n be odd. Prove that $|A| = 0$.

1.2. Prove that the determinant of a skew-symmetric matrix of even order does not change if to all its elements we add the same number.

1.3. Compute the determinant of a skew-symmetric matrix A_n of order $2n$ with each element above the main diagonal being equal to 1.

1.4. Prove that for $n \geq 3$ the terms in the expansion of a determinant of order n cannot be all positive.

1.5. Let $a_{ij} = a^{|i-j|}$. Compute $|a_{ij}|_1^n$.

1.6. Let $\Delta_3 = \begin{vmatrix} 1 & -1 & 0 & 0 \\ x & h & -1 & 0 \\ x^2 & hx & h & -1 \\ x^3 & hx^2 & hx & h \end{vmatrix}$ and define Δ_n accordingly. Prove that $\Delta_n = (x+h)^n$.

1.7. Compute $|c_{ij}|_1^n$, where $c_{ij} = a_i b_j$ for $i \neq j$ and $c_{ii} = x_i$.

1.8. Let $a_{i,i+1} = c_i$ for $i = 1, \dots, n$, the other matrix elements being zero. Prove that the determinant of the matrix $I + A + A^2 + \dots + A^{n-1}$ is equal to $(1-c)^{n-1}$, where $c = c_1 \dots c_n$.

1.9. Compute $|a_{ij}|_1^n$, where $a_{ij} = (1 - x_i y_j)^{-1}$.

1.10. Let $a_{ij} = \binom{n+i}{j}$. Prove that $|a_{ij}|_0^m = 1$.

1.11. Prove that for any real numbers a, b, c, d, e and f

$$\begin{vmatrix} (a+b)de-(d+e)ab & ab-de & a+b-d-e \\ (b+c)ef-(e+f)bc & bc-ef & b+c-e-f \\ (c+d)fa-(f+a)cd & cd-fa & c+d-f-a \end{vmatrix} = 0.$$

Vandermonde's determinant.

1.12. Compute

$$\begin{vmatrix} 1 & x_1 & \dots & x_1^{n-2} & (x_2+x_3+\dots+x_n)^{n-1} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & x_n & \dots & x_n^{n-2} & (x_1+x_2+\dots+x_{n-1})^{n-1} \end{vmatrix}.$$

1.13. Compute

$$\begin{vmatrix} 1 & x_1 & \dots & x_1^{n-2} & x_2 x_3 \dots x_n \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & x_n & \dots & x_n^{n-2} & x_1 x_2 \dots x_{n-1} \end{vmatrix}.$$

1.14. Compute $|a_{ik}|_0^n$, where $a_{ik} = \lambda_i^{n-k}(1+\lambda_i^2)^k$.

1.15. Let $V = \left\|a_{ij}\right\|_0^n$, where $a_{ij} = x_i^{j-1}$, be a Vandermonde matrix; let V_k be the matrix obtained from V by deleting its $(k+1)$st column (which consists of the kth powers) and adding instead the nth column consisting of the nth powers. Prove that

$$\det V_k = \sigma_{n-k}(x_1, \dots, x_n) \det V.$$

1.16. Let $a_{ij} = \binom{in}{j}$. Prove that $|a_{ij}|_1^r = n^{r(r+1)/2}$ for $r \leq n$.

1.17. Given $k_1, \dots, k_n \in \mathbb{Z}$, compute $|a_{ij}|_1^n$, where

$$a_{i,j} = \begin{cases} \dfrac{1}{(k_i+j-i)!} & \text{for } k_i+j-i \geq 0 \ , \\ a_{ij} = 0 & \text{for } k_i+j-i < 0. \end{cases}$$

1.18. Let $s_k = p_1 x_1^k + \dots + p_n x_n^k$, and $a_{i,j} = s_{i+j}$. Prove that

$$|a_{ij}|_0^{n-1} = p_1 \dots p_n \prod_{i>j} (x_i - x_j)^2.$$

1.19. Let $s_k = x_1^k + \dots + x_n^k$. Compute

$$\begin{vmatrix} s_0 & s_1 & \dots & s_{n-1} & 1 \\ s_1 & s_2 & \dots & s_n & y \\ \vdots & \vdots & \dots & \vdots & \vdots \\ s_n & s_{n+1} & \dots & s_{2n-1} & y^n \end{vmatrix}.$$

1.20. Let $a_{ij} = (x_i + y_j)^n$. Prove that

$$|a_{ij}|_0^n = \binom{n}{1} \dots \binom{n}{n} \cdot \prod_{i>k} (x_i - x_k)(y_k - y_i).$$

1.21. Find all solutions of the system

$$\begin{cases} \lambda_1 + \dots + \lambda_n = 0 \\ \dots\dots\dots\dots \\ \lambda_1^n + \dots + \lambda_n^n = 0 \end{cases}$$

in $\mathbb{C}$.

1.22. Let $\sigma_k(x_0, \dots, x_n)$ be the kth elementary symmetric function. Set: $\sigma_0 = 1$, $\sigma_k(\widehat{x}_i) = \sigma_k(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. Prove that if $a_{ij} = \sigma_i(\widehat{x}_j)$ then $|a_{ij}|_0^n = \prod_{i<j}(x_i - x_j)$.

Relations among determinants.

1.23. Let $b_{ij} = (-1)^{i+j} a_{ij}$. Prove that $|a_{ij}|_1^n = |b_{ij}|_1^n$.

1.24. Prove that

$$\begin{vmatrix} a_1c_1 & a_2d_1 & a_1c_2 & a_2d_2 \\ a_3c_1 & a_4d_1 & a_3c_2 & a_4d_2 \\ b_1c_3 & b_2d_3 & b_1c_4 & b_2d_4 \\ b_3c_3 & b_4d_3 & b_3c_4 & b_4d_4 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} \cdot \begin{vmatrix} b_1 & b_2 \\ b_3 & b_4 \end{vmatrix} \cdot \begin{vmatrix} c_1 & c_2 \\ c_3 & c_4 \end{vmatrix} \cdot \begin{vmatrix} d_1 & d_2 \\ d_3 & d_4 \end{vmatrix}.$$

1.25. Prove that

$$\begin{vmatrix} a_1 & 0 & 0 & b_1 & 0 & 0 \\ 0 & a_2 & 0 & 0 & b_2 & 0 \\ 0 & 0 & a_3 & 0 & 0 & b_3 \\ b_{11} & b_{12} & b_{13} & a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} & a_{21} & a_{22} & a_{23} \\ b_{31} & b_{32} & b_{33} & a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_1a_{11} - b_1b_{11} & a_2a_{12} - b_2b_{12} & a_3a_{13} - b_3b_{13} \\ a_1a_{21} - b_1b_{21} & a_2a_{22} - b_2b_{22} & a_3a_{23} - b_3b_{23} \\ a_1a_{31} - b_1b_{31} & a_2a_{32} - b_2b_{32} & a_3a_{33} - b_3b_{33} \end{vmatrix}.$$

1.26. Let $s_k = \sum_{i=1}^n a_{ki}$. Prove that

$$\begin{vmatrix} s_1 - a_{11} & \dots & s_1 - a_{1n} \\ \vdots & \dots & \vdots \\ s_n - a_{n1} & \dots & s_n - a_{nn} \end{vmatrix} = (-1)^{n-1}(n-1) \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \dots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}.$$

1.27. Prove that

$$\begin{vmatrix} \binom{n}{m_1} & \binom{n}{m_1-1} & \dots & \binom{n}{m_1-k} \\ \vdots & \vdots & \dots & \vdots \\ \binom{n}{m_k} & \binom{n}{m_k-1} & \dots & \binom{n}{m_k-k} \end{vmatrix} = \begin{vmatrix} \binom{n}{m_1} & \binom{n+1}{m_1} & \dots & \binom{n+k}{m_1} \\ \vdots & \vdots & \dots & \vdots \\ \binom{n}{m_k} & \binom{n+1}{m_k} & \dots & \binom{n+k}{m_k} \end{vmatrix}.$$

1.28. Let $\Delta_n(k) = |a_{ij}|_0^n$, where $a_{ij} = \binom{k+i}{2j}$. Prove that

$$\Delta_n(k) = \frac{k(k+1)\dots(k+n-1)}{1 \cdot 3 \dots (2n-1)} \Delta_{n-1}(k-1).$$

1.29. Let $D_n = |a_{ij}|_0^n$, where $a_{ij} = \binom{n+i}{2j-1}$. Prove that $D_n = 2^{n(n+1)/2}$.

1.30. Given numbers $a_0, a_1, ..., a_{2n}$, let $b_k = \sum_{i=0}^k (-1)^i \binom{k}{i} a_i \quad (k = 0, \dots, 2n)$; let $a_{ij} = a_{i+j}$, and $b_{ij} = b_{i+j}$. Prove that $|a_{ij}|_0^n = |b_{ij}|_0^n$.

1.31. Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ and $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$, where A_{11} and B_{11}, and also A_{22} and B_{22}, are square matrices of the same size such that $\operatorname{rank} A_{11} = \operatorname{rank} A$ and $\operatorname{rank} B_{11} = \operatorname{rank} B$. Prove that

$$\begin{vmatrix} A_{11} & B_{12} \\ A_{21} & B_{22} \end{vmatrix} \cdot \begin{vmatrix} A_{11} & A_{12} \\ B_{21} & B_{22} \end{vmatrix} = |A + B| \cdot |A_{11}| \cdot |B_{22}|.$$

1.32. Let A and B be square matrices of order n. Prove that $|A| \cdot |B| = \sum_{k=1}^n |A_k| \cdot |B_k|$, where the matrices A_k and B_k are obtained from A and B, respectively, by interchanging the respective first and kth columns, i.e., the first column of A is replaced with the kth column of B and the kth column of B is replaced with the first column of A.

2. Minors and cofactors

2.1. There are many instances when it is convenient to consider the determinant of the matrix whose elements stand at the intersection of certain p rows and p columns of a given matrix A. Such a determinant is called a *pth order minor of A*.

For convenience we introduce the following notation:

$$A\begin{pmatrix} i_1 & \dots i_p \\ k_1 & \dots k_p \end{pmatrix} = \begin{vmatrix} a_{i_1k_1} & a_{i_1k_2} & \dots & a_{i_1k_p} \\ \vdots & \vdots & \dots & \vdots \\ a_{i_pk_1} & a_{i_pk_2} & \dots & a_{i_pk_p} \end{vmatrix}.$$

If $i_1 = k_1, \dots, i_p = k_p$, the minor is called a *principal* one.

2.2. A nonzero minor of the maximal order is called a *basic minor* and its order is called the *rank* of the matrix.

Theorem. *If $A\binom{i_1\dots i_p}{k_1\dots k_p}$ is a basic minor of a matrix A, then the rows of A are linear combinations of rows numbered $i_1,\dots,i_p$ and these rows are linearly independent.*

Proof. The linear independence of the rows numbered $i_1,\dots,i_p$ is obvious since the determinant of a matrix with linearly dependent rows vanishes.

The cases when the size of A is $m\times p$ or $p\times m$ are also clear.

It suffices to carry out the proof for the minor $A\binom{1\ \dots p}{1\ \dots p}$. The determinant

$$\begin{vmatrix} a_{11} & \dots & a_{1p} & a_{1j} \\ \vdots & \dots & \vdots & \vdots \\ a_{p1} & \dots & a_{pp} & a_{pj} \\ a_{i1} & \dots & a_{ip} & a_{ij} \end{vmatrix}$$

vanishes for $j\leq p$ as well as for $j>p$. Its expansion with respect to the last column is a relation of the form

$$a_{1j}c_1+a_{2j}c_2+\dots+a_{pj}c_p+a_{ij}c=0,$$

where the numbers $c_1,\ \dots\ ,\ c_p,\ c$ do not depend on j (but depend on i) and $c=A\binom{1\ \dots p}{1\ \dots p}\neq 0$. Hence, the ith row is equal to the linear combination of the first p rows with the coefficients $\dfrac{-c_1}{c},\ \dots\ ,\ \dfrac{-c_p}{c}$, respectively. $\square$

2.2.1. Corollary. *If $A\binom{i_1\ \dots i_p}{k_1\ \dots k_p}$ is a basic minor then all rows of A belong to the linear space spanned by the rows numbered $i_1,\dots,i_p$; therefore, the rank of A is equal to the maximal number of its linearly independent rows.*

2.2.2. Corollary. *The rank of a matrix is also equal to the maximal number of its linearly independent columns.*

2.3. Theorem (The Binet-Cauchy formula). *Let A and B be matrices of size $n\times m$ and $m\times n$, respectively, and $n\leq m$. Then*

$$\det AB=\sum_{1\leq k_1<k_2<\dots<k_n\leq m} A_{k_1\dots k_n}B^{k_1\dots k_n},$$

where $A_{k_1\dots k_n}$ is the minor obtained from the columns of A whose numbers are $k_1,\dots,k_n$ and $B^{k_1\dots k_n}$ is the minor obtained from the rows of B whose numbers are $k_1,\dots,k_n$.

Proof. Let $C=AB$, $c_{ij}=\sum_{k=1}^m a_{ik}b_{ki}$. Then

$$\begin{aligned}\det C&=\sum_\sigma(-1)^\sigma\sum_{k_1}a_{1k_1}b_{k_1\sigma(1)}\cdots\sum_{k_n}b_{k_n\sigma(n)}\\ &=\sum_{k_1,\dots,k_n=1}^m a_{1k_1}\dots a_{nk_n}\sum_\sigma(-1)^\sigma b_{k_1\sigma(1)}\dots b_{k_n\sigma(n)}\\ &=\sum_{k_1,\dots,k_n=1}^m a_{1k_1}\dots a_{nk_n}B^{k_1\dots k_n}.\end{aligned}$$

The minor $B^{k_1\dots k_n}$ is nonzero only if the numbers $k_1,\dots,k_n$ are distinct; therefore, the summation can be performed over distinct numbers $k_1,\dots,k_n$. Since $B^{\tau(k_1)\dots\tau(k_n)} = (-1)^\tau B^{k_1\dots k_n}$ for any permutation τ of the numbers $k_1,\dots,k_n$, then

$$\begin{aligned}\sum_{k_1,\dots,k_n=1}^{m} a_{1k_1}\dots a_{nk_n}B^{k_1\dots k_n} &= \sum_{k_1<k_2<\dots<k_n}(-1)^\tau a_{1\tau(1)}\dots a_{n\tau(n)}B^{k_1\dots k_n}\\ &= \sum_{1\le k_1<k_2<\dots<k_n\le m} A_{k_1\dots k_n}B^{k_1\dots k_n}. \quad \square\end{aligned}$$

Remark. Another proof is given in the solution of Problem 28.7

2.4. Recall the formula for expansion of the determinant of a matrix with respect to its ith row:

$$|a_{ij}|_1^n = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}M_{ij}, \tag{1}$$

where M_{ij} is the determinant of the matrix obtained from the matrix $A = \|a_{ij}\|_1^n$ by deleting its ith row and jth column. The number $A_{ij} = (-1)^{i+j}M_{ij}$ is called the *cofactor of the element* a_{ij} *in* A.

It is possible to expand a determinant not only with respect to one row, but also with respect to several rows simultaneously.

Fix rows numbered $i_1,\dots,i_p$, where $i_1 < i_2 < \dots < i_p$. In the expansion of the determinant of A there occur products of terms of the expansion of the minor $A\binom{i_1\ \dots i_p}{j_1\ \dots j_p}$ by terms of the expansion of the minor $A\binom{i_{p+1}\ \dots i_n}{j_{p+1}\ \dots j_n}$, where $j_1 < \dots < j_p$; $i_{p+1} < \dots < i_n$; $j_{p+1} < \dots < j_n$ and there are no other terms in the expansion of the determinant of A.

To compute the signs of these products let us shuffle the rows and the columns so as to place the minor $A\binom{i_1\ \dots i_p}{j_1\ \dots j_p}$ in the upper left corner. To this end we have to perform

$$(i_1 - 1) + \dots + (i_p - p) + (j_1 - 1) + \dots + (j_p - p) \equiv i + j \pmod 2$$

permutations, where $i = i_1 + \dots + i_p$, $j = j_1 + \dots + j_p$.

The number $(-1)^{i+j}A\binom{i_{p+1}\ \dots i_n}{j_{p+1}\ \dots j_n}$ is called the *cofactor of the minor* $A\binom{i_1\ \dots i_p}{j_1\ \dots j_p}$.

We have proved the following statement:

2.4.1. Theorem (Laplace). *Fix p rows of the matrix A. Then the sum of products of the minors of order p that belong to these rows by their cofactors is equal to the determinant of A.*

The matrix $\operatorname{adj} A = (A_{ij})^T$ is called the *(classical) adjoint*[1] of A. Let us prove that $A\cdot(\operatorname{adj} A) = |A|\cdot I$. To this end let us verify that $\sum_{j=1}^{n} a_{ij}A_{kj} = \delta_{ki}|A|$.

For $k = i$ this formula coincides with (1). If $k \neq i$, replace the kth row of A with the ith one. The determinant of the resulting matrix vanishes; its expansion with respect to the kth row results in the desired identity:

$$0 = \sum_{j=1}^{n} a'_{kj}A_{kj} = \sum_{j=1}^{n} a_{ij}A_{kj}.$$

[1] We will briefly write *adjoint* instead of the *classical adjoint*.

If A is invertible then $A^{-1} = \dfrac{\operatorname{adj} A}{|A|}$.

2.4.2. Theorem. *The operation* adj *has the following properties:*
a) $\operatorname{adj} AB = \operatorname{adj} B \cdot \operatorname{adj} A$;
b) $\operatorname{adj} XAX^{-1} = X(\operatorname{adj} A)X^{-1}$;
c) *if* $AB = BA$ *then* $(\operatorname{adj} A)B = B(\operatorname{adj} A)$.

Proof. If A and B are invertible matrices, then $(AB)^{-1} = B^{-1}A^{-1}$. Since for an invertible matrix A we have $\operatorname{adj} A = A^{-1}|A|$, headings a) and b) are obvious. Let us consider heading c).

If $AB = BA$ and A is invertible, then

$$A^{-1}B = A^{-1}(BA)A^{-1} = A^{-1}(AB)A^{-1} = BA^{-1}.$$

Therefore, for invertible matrices the theorem is obvious.

In each of the equations a) – c) both sides continuously depend on the elements of A and B. Any matrix A can be approximated by matrices of the form $A_\varepsilon = A + \varepsilon I$ which are invertible for sufficiently small nonzero ε. (Actually, if $a_1, \dots, a_r$ is the whole set of eigenvalues of A, then A_ε is invertible for all $\varepsilon \neq -a_i$.) Besides, if $AB = BA$, then $A_\varepsilon B = BA_\varepsilon$. $\square$

2.5. The relations between the minors of a matrix A and the complementary to them minors of the matrix $(\operatorname{adj} A)^T$ are rather simple.

2.5.1. Theorem. *Let* $A = \left\|a_{ij}\right\|_1^n$, $(\operatorname{adj} A)^T = |A_{ij}|_1^n$, $1 \le p < n$. *Then*

$$\begin{vmatrix} A_{11} & \dots & A_{1p} \\ \vdots & \dots & \vdots \\ A_{p1} & \dots & A_{pp} \end{vmatrix} = |A|^{p-1} \begin{vmatrix} a_{p+1,p+1} & \dots & a_{p+1,n} \\ \vdots & \dots & \vdots \\ a_{n,p+1} & \dots & a_{nn} \end{vmatrix}.$$

Proof. For $p = 1$ the statement coincides with the definition of the cofactor A_{11}. Let $p > 1$. Then the identity

$$\begin{pmatrix} A_{11} & \dots & A_{1p} & A_{1,p+1} & \dots & A_{1n} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ A_{p1} & \dots & A_{pp} & A_{p,p+1} & \dots & A_{pn} \\ & 0 & & & I & \end{pmatrix} \begin{pmatrix} a_{11} & \dots & a_{n1} \\ \vdots & \dots & \vdots \\ a_{1n} & \dots & a_{nn} \end{pmatrix}$$

$$= \begin{vmatrix} |A| & & 0 & & \\ & \dots & & & 0 \\ 0 & & |A| & & \\ a_{1,p+1} & \dots & & \dots & a_{n,p+1} \\ \vdots & \dots & & \dots & \vdots \\ a_{1n} & \dots & & \dots & a_{nn} \end{vmatrix}.$$

implies that

$$\begin{vmatrix} A_{11} & \dots & A_{1p} \\ \vdots & \dots & \vdots \\ A_{p1} & \dots & A_{pp} \end{vmatrix} \cdot |A| = |A|^p \cdot \begin{vmatrix} a_{p+1,p+1} & \dots & a_{p+1,n} \\ \vdots & \dots & \vdots \\ a_{n,p+1} & \dots & a_{nn} \end{vmatrix}.$$

If $|A| \neq 0$, then dividing by $|A|$ we get the desired conclusion. For $|A| = 0$ the statement follows from the continuity of the both parts of the desired identity with respect to a_{ij}. $\square$

COROLLARY. *If A is not invertible then* $\operatorname{rank}(\operatorname{adj} A) \leq 1$.

PROOF. For $p = 2$ we get

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = |A| \cdot \begin{vmatrix} a_{33} & \dots & a_{3n} \\ \vdots & \dots & \vdots \\ a_{n3} & \dots & a_{nn} \end{vmatrix} = 0.$$

Besides, the transposition of any two rows of the matrix A induces the same transposition of the columns of the adjoint matrix and all elements of the adjoint matrix change sign (look what happens with the determinant of A and with the matrix A^{-1} for an invertible A under such a transposition). $\square$

Application of transpositions of rows and columns makes it possible for us to formulate Theorem 2.5.1 in the following more general form.

2.5.2. THEOREM (Jacobi). *Let $A = \left\|a_{ij}\right\|_1^n$, $(\operatorname{adj} A)^T = \left\|A_{ij}\right\|_1^n$, $1 \leq p < n$, $\sigma = \begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix}$ an arbitrary permutation. Then*

$$\begin{vmatrix} A_{i_1 j_1} & \dots & A_{i_1 j_p} \\ \vdots & \dots & \vdots \\ A_{i_p j_1} & \dots & A_{i_p j_p} \end{vmatrix} = (-1)^\sigma \begin{vmatrix} a_{i_{p+1}, j_{p+1}} & \dots & a_{i_{p+1}, j_n} \\ \vdots & \dots & \vdots \\ a_{i_n, j_{p+1}} & \dots & a_{i_n, j_n} \end{vmatrix} \cdot |A|^{p-1}.$$

PROOF. Let us consider matrix $B = \left\|b_{kl}\right\|_1^n$, where $b_{kl} = a_{i_k j_l}$. It is clear that $|B| = (-1)^\sigma |A|$. Since a transposition of any two rows (resp. columns) of A induces the same transposition of the columns (resp. rows) of the adjoint matrix and all elements of the adjoint matrix change their sings, $B_{kl} = (-1)^\sigma A_{i_k j_l}$.

Applying Theorem 2.5.1 to matrix B we get

$$\begin{vmatrix} (-1)^\sigma A_{i_1 j_1} & \dots & (-1)^\sigma A_{i_1 j_p} \\ \vdots & \dots & \vdots \\ (-1)^\sigma A_{i_p j_1} & \dots & (-1)^\sigma A_{i_p j_p} \end{vmatrix} = ((-1)^\sigma)^{p-1} \begin{vmatrix} a_{i_{p+1}, j_{p+1}} & \dots & a_{i_{p+1}, j_n} \\ \vdots & \dots & \vdots \\ a_{i_n, j_{p+1}} & \dots & a_{i_n, j_n} \end{vmatrix}.$$

By dividing the both parts of this equality by $((-1)^\sigma)^p$ we obtain the desired. $\square$

2.6. In addition to the adjoint matrix of A it is sometimes convenient to consider the *compound matrix* $\left\|M_{ij}\right\|_1^n$ consisting of the $(n-1)$st order minors of A. The determinant of the adjoint matrix is equal to the determinant of the compound one (see, e.g., Problem 1.23).

For a matrix A of size $m \times n$ we can also consider a matrix whose elements are rth order minors $A\begin{pmatrix} i_1 & \dots & i_r \\ j_1 & \dots & j_r \end{pmatrix}$, where $r \leq \min(m, n)$. The resulting matrix

$C_r(A)$ is called the *rth compound matrix of* A. For example, if $m = n = 3$ and $r = 2$, then

$$C_2(A) = \begin{pmatrix} A\begin{pmatrix}12\\12\end{pmatrix} & A\begin{pmatrix}12\\13\end{pmatrix} & A\begin{pmatrix}12\\23\end{pmatrix} \\ A\begin{pmatrix}13\\12\end{pmatrix} & A\begin{pmatrix}13\\13\end{pmatrix} & A\begin{pmatrix}13\\23\end{pmatrix} \\ A\begin{pmatrix}23\\12\end{pmatrix} & A\begin{pmatrix}23\\13\end{pmatrix} & A\begin{pmatrix}23\\23\end{pmatrix} \end{pmatrix}.$$

Making use of Binet–Cauchy's formula we can show that $C_r(AB) = C_r(A)C_r(B)$. For a square matrix A of order n we have the Sylvester identity

$$\det C_r(A) = (\det A)^p, \text{ where } p = \binom{n-1}{r-1}.$$

The simplest proof of this statement makes use of the notion of exterior power (see Theorem 28.5.3).

2.7. Let $1 \le m \le r < n$, $A = \|a_{ij}\|_1^n$. Set $A_n = |a_{ij}|_1^n$, $A_m = |a_{ij}|_1^m$. Consider the matrix $S^r_{m,n}$ whose elements are the rth order minors of A containing the left upper corner principal minor A_m. The determinant of $S^r_{m,n}$ is a minor of order $\binom{n-m}{r-m}$ of $C_r(A)$. The determinant of $S^r_{m,n}$ can be expressed in terms of A_m and A_n.

Theorem (Generalized Sylvester's identity, [Mohr,1953]).

$$(1) \qquad |S^r_{m,n}| = A_m^p A_n^q, \text{ where } p = \binom{n-m-1}{r-m}, q = \binom{n-m-1}{r-m-1}.$$

Proof. Let us prove identity (1) by induction on n. For $n = 2$ it is obvious.

The matrix $S^r_{0,n}$ coincides with $C_r(A)$ and since $|C_r(A)| = A_n^q$, where $q = \binom{n-1}{r-1}$ (see Theorem 28.5.3), then (1) holds for $m = 0$ (we assume that $A_0 = 1$). Both sides of (1) are continuous with respect to a_{ij} and, therefore, it suffices to prove the inductive step when $a_{11} \ne 0$.

All minors considered contain the first row and, therefore, from the rows whose numbers are $2, \dots, n$ we can subtract the first row multiplied by an arbitrary factor; this operation does not affect $\det(S^r_{m,n})$. With the help of this operation all elements of the first column of A except a_{11} can be made equal to zero. Let $\overline{A}$ be the matrix obtained from the new one by strikinging out the first column and the first row, and let $\overline{S}^{r-1}_{m-1,n-1}$ be the matrix composed of the minors of order $r-1$ of $\overline{A}$ containing its left upper corner principal minor of order $m-1$.

Obviously, $S^r_{m,n} = a_{11}\overline{S}^{r-1}_{m-1,n-1}$ and we can apply to $\overline{S}^{r-1}_{m-1,n-1}$ the inductive hypothesis (the case $m-1=0$ was considered separately). Besides, if $\overline{A}_{m-1}$ and $\overline{A}_{n-1}$ are the left upper corner principal minors of orders $m-1$ and $n-1$ of A, respectively, then $A_m = a_{11}\overline{A}_{m-1}$ and $A_n = a_{11}\overline{A}_{n-1}$. Therefore,

$$|S^r_{m,n}| = a_{11}^t \overline{A}_{m-1}^{p_1} \overline{A}_{n-1}^{q_1} = a_{11}^{t-p_1-q_1} A_m^{p_1} A_n^{q_1},$$

where $t = \binom{n-m}{r-m}$, $p_1 = \binom{n-m-1}{r-m} = p$ and $q_1 = \binom{n-m-1}{r-m-1} = q$. Taking into account that $t = p + q$, we get the desired conclusion. □

Remark. Sometimes the term "Sylvester's identity" is applied to identity (1) not only for $m = 0$ but also for $r = m + 1$, i.e., $|S^{m+1}_{m,n}| = A_m^{n-m} A_n$

2.8 THEOREM (Chebotarev). *Let p be a prime and $\varepsilon = \exp(2\pi i/p)$. Then all minors of the Vandermonde matrix $\left\|a_{ij}\right\|_0^{p-1}$, where $a_{ij} = \varepsilon^{ij}$, are nonzero.*

PROOF (Following [Reshetnyak, 1955]). Suppose that

$$\begin{vmatrix} \varepsilon^{k_1 l_1} & \dots & \varepsilon^{k_1 l_j} \\ \varepsilon^{k_2 l_1} & \dots & \varepsilon^{k_2 l_j} \\ \vdots & \dots & \vdots \\ \varepsilon^{k_j l_1} & \dots & \varepsilon^{k_j l_j} \end{vmatrix} = 0.$$

Then there exist complex numbers $c_1, \dots, c_j$ not all equal to 0 such that the linear combination of the corresponding columns with coefficients $c_1, \dots, c_j$ vanishes, i.e., the numbers $\varepsilon^{k_1}, \dots, \varepsilon^{k_j}$ are roots of the polynomial $c_1 x^{l_1} + \dots + c_j x^{l_j}$. Let

$$(1) \qquad (x - \varepsilon^{k_1}) \dots (x - \varepsilon^{k_j}) = x^j - b_1 x^{j-1} + \dots \pm b_j.$$

Then

$$(2) \qquad c_1 x^{l_1} + \dots + c_j x^{l_j} = (b_0 x^j - b_1 x^{j-1} + \dots \pm b_j)(a_s x^s + \dots + a_0),$$

where $b_0 = 1$ and $a_s \neq 0$. For convenience let us assume that $b_t = 0$ for $t > j$ and $t < 0$. The coefficient of x^{j+s-t} in the right-hand side of (2) is equal to $\pm(a_s b_t - a_{s-1} b_{t-1} + \dots \pm a_0 b_{t-s})$. The degree of the polynomial (2) is equal to $s + j$ and it is only the coefficients of the monomials of degrees $l_1, \dots, l_j$ that may be nonzero and, therefore, there are $s + 1$ zero coefficients:

$$a_s b_t - a_{s-1} b_{t-1} + \dots \pm a_0 b_{t-s} = 0 \quad \text{for } t = t_0, t_1, \dots, t_s$$

The numbers $a_0, \dots, a_{s-1}, a_s$ are not all zero and therefore, $|c_{kl}|_0^s = 0$ for $c_{kl} = b_t$, where $t = t_k - l$.

Formula (1) shows that b_t can be represented in the form $f_t(\varepsilon)$, where f_t is a polynomial with integer coefficients and this polynomial is the sum of $\binom{j}{t}$ powers of ε; hence, $f_t(1) = \binom{j}{t}$. Since $c_{kl} = b_t = f_t(\varepsilon)$, then $|c_{kl}|_0^s = g(\varepsilon)$ and $g(1) = |c'_{kl}|_0^s$, where $c'_{kl} = \binom{j}{t_k - l}$. The polynomial $q(x) = x^{p-1} + \dots + x + 1$ is irreducible over $\mathbb{Z}$ (see Appendix 2) and $q(\varepsilon) = 0$. Therefore, $g(x) = q(x)\varphi(x)$, where φ is a polynomial with integer coefficients (see Appendix 1). Therefore, $g(1) = q(1)\varphi(1) = p\varphi(1)$, i.e., $g(1)$ is divisible by p.

To get a contradiction it suffices to show that the number $g(1) = |c'_{kl}|_0^s$, where $c'_{kl} = \binom{j}{t_k - l}$, $0 \leq t_k \leq j + s$ and $0 < j + s \leq p - 1$, is not divisible by p. It is easy to verify that $\Delta = |c'_{kl}|_0^s = |a_{kl}|_0^s$, where $a_{kl} = \binom{j+l}{t_k}$ (see Problem 1.27). It is also clear that

$$\binom{j+l}{t} = \left(1 - \frac{t}{j+l+1}\right) \dots \left(1 - \frac{t}{j+s}\right) \binom{j+s}{t} = \varphi_{s-l}(t) \binom{j+s}{t}.$$

Hence,

$$\Delta = \prod_{\lambda=0}^{s} \binom{j+s}{t_\lambda} \begin{vmatrix} \varphi_s(t_0) & \varphi_{s-1}(t_0) & \dots & 1 \\ \varphi_s(t_1) & \varphi_{s-1}(t_1) & \dots & 1 \\ \vdots & \vdots & \dots & \vdots \\ \varphi_s(t_s) & \varphi_{s-1}(t_s) & \dots & 1 \end{vmatrix} = \pm \prod_{\lambda=0}^{s} \left(\binom{j+s}{t_\lambda} A_\lambda \right) \prod_{\mu > \nu} (t_\mu - t_\nu),$$

where $A_0, A_1, \dots, A_s$ are the coefficients of the highest powers of t in the polynomials $\varphi_0(t), \varphi_1(t), \dots, \varphi_s(t)$, respectively, where $\varphi_0(t) = 1$; the degree of $\varphi_i(t)$ is equal to i. Clearly, the product obtained has no irreducible fractions with numerators divisible by p, because $j + s < p$. □

Problems

2.1. Let A_n be a matrix of size $n\times n$. Prove that $|A+\lambda I| = \lambda^n + \sum_{k=1}^n S_k\lambda^{n-k}$, where S_k is the sum of all $\binom{n}{k}$ principal kth order minors of A.

2.2. Prove that

$$\begin{vmatrix} a_{11} & \dots & a_{1n} & x_1 \\ \vdots & \dots & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} & x_n \\ y_1 & \dots & y_n & 0 \end{vmatrix} = -\sum_{i,j} x_i y_j A_{ij},$$

where A_{ij} is the cofactor of a_{ij} in $\left\|a_{ij}\right\|_1^n$.

2.3. Prove that the sum of principal k-minors of A^TA is equal to the sum of squares of all k-minors of A.

2.4. Prove that

$$\begin{vmatrix} u_1a_{11} & \dots & u_na_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & \dots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} + \dots + \begin{vmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & \dots & \vdots \\ u_1a_{n1} & \dots & u_na_{nn} \end{vmatrix} = (u_1+\dots+u_n)|A|.$$

Inverse and adjoint matrices

2.5. Let A and B be square matrices of order n. Compute

$$\begin{pmatrix} I & A & C \\ 0 & I & B \\ 0 & 0 & I \end{pmatrix}^{-1}.$$

2.6. Prove that the matrix inverse to an invertible upper triangular matrix is also an upper triangular one.

2.7. Give an example of a matrix of order n whose adjoint has only one nonzero element and this element is situated in the ith row and jth column for given i and j.

2.8. Let x and y be columns of length n. Prove that

$$\operatorname{adj}(I - xy^T) = xy^T + (1 - y^Tx)I.$$

2.9. Let A be a skew-symmetric matrix of order n. Prove that $\operatorname{adj} A$ is a symmetric matrix for odd n and a skew-symmetric one for even n.

2.10. Let A_n be a skew-symmetric matrix of order n with elements $+1$ above the main diagonal. Calculate $\operatorname{adj} A_n$.

2.11. The matrix $\operatorname{adj}(A-\lambda I)$ can be expressed in the form $\sum_{k=0}^{n-1}\lambda^k A_k$, where n is the order of A. Prove that:

a) for any k $(1\le k\le n-1)$ the matrix $A_kA - A_{k-1}$ is a scalar matrix;

b) the matrix A_{n-s} can be expressed as a polynomial of degree $s-1$ in A.

2.12. Find all matrices A with nonnegative elements such that all elements of A^{-1} are also nonnegative.

2.13. Let $\varepsilon = \exp(2\pi i/n)$; $A = \left\|a_{ij}\right\|_1^n$, where $a_{ij} = \varepsilon^{ij}$. Calculate the matrix A^{-1}.

2.14. Calculate the matrix inverse to the Vandermonde matrix V.