# Stokes's Theorem, Data, and the Polar Ice Caps

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**Abstract.** Geographers and climate scientists alike sometimes need to estimate the area of a large region on the surface of the Earth, such as the polar ice caps. Doing so using only a sequence of latitude-longitude data points along a piecewise-linear approximation of the boundary of the region can be accomplished via a novel use of Stokes's theorem that generalizes classical and contemporary applications of Green's Theorem to data. The polar ice caps are seen to hide a complication that makes their estimation from data singularly interesting.

**1. STOKES'S THEOREM.** The calculus student's long ascent through multivariable calculus usually culminates in an encounter with three mathematical names: Green, Gauss, and Stokes. Their eponymous theorems mean for most students of calculus the journey's end, with a quick memorization of relevant formulae. A smaller number of students are led to some of the applications for which these theorems were first used. These largely concern electromagnetics (say, Maxwell's equations [**5, 6, 8**]) or fluid dynamics (the theorems of Kelvin and Helmholtz [**3, 6**]). These applications in field physics are more than a century old. Examples of a more modern nature from statistics, data analysis, economics, or other fields, are either not well known or too advanced for a first-year undergraduate course. The net result of this is too often an unmotivated, abrupt end to multivariable calculus.

A very few students are permitted a flash of mathematical beauty beyond the traditional physics-centered approach of the classical calculus texts. This is best done via differential forms, which, if presented in a simplified Euclidean setting, is entirely reasonable for first-year students (several texts now include such material [**4, 6, 8**]). One may introduce the basis $k$-forms on $\mathbb{R}^n$, $dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$, in terms of oriented projected $k$-dimensional volume via determinants. With a few algebraic rules concerning $\wedge$, $d$, and integration, one simplifies the theorems of Green, Gauss, and Stokes into the full version of Stokes's theorem:

$$\int_{\partial D} \alpha = \int_D d\alpha \,, \tag{1}$$

for $D$ a smooth $(k+1)$-dimensional domain in $\mathbb{R}^n$ with $k$-dimensional boundary $\partial D$ and $\alpha$ a $k$-form on $D$. A bit of dexterity is required to not prompt too many awkward questions on what a *smooth domain* means. This theorem is unifying and beautiful, but does not seem to lend itself to many applications of a concrete nature.

**2. COMPUTING EUCLIDEAN AREA AND VOLUME.** There is one popular application of Green's theorem that has an interpretation in terms of data and approximation. Consider, for example, what happens when an ultrasound technician performs a scan of a partially obstructed artery. A cross-section of the artery is viewed with free-flowing and obstructed regions visible. The cross-sectional area of the free-flowing region may be specified by the operator marking a sequence of points on the image as ordered landmarks for an approximate boundary (Figure 1[left]). This free cross-section may or may not be convex.

How is the area estimated? One well-known method for approximating area invokes Green's theorem. Using planar coordinates $(x, y)$, the area $A$ of the free cross-section $D$ can be computed as a loop integral:

$$A = \int_D dx \wedge dy = \frac{1}{2} \oint_{\partial D} x \, dy - y \, dx \, . \tag{2}$$

By approximating the boundary $\partial D$ as a piecewise-linear curve of straight segments $\gamma^i$ from $(x_1^i, y_1^i)$ to $(x_2^i, y_2^i)$, and integrating the 1-form above on each such segment for $i = 1$ to $N$, one obtains the classic combinatorial formula for area [9],

$$A \approx \frac{1}{2} \sum_{i=1}^{N} x_1^i y_2^i - x_2^i y_1^i \, . \tag{3}$$

This has applications ranging from medical image data to determining area of counties and plots of land. The curious reader may wish to show that other geometric quantities (centroid coordinates and moments of inertia, for example) can be likewise approximated via combinatorial formulae, with a simple modification to the 1-form integrated over the straight-line segments.
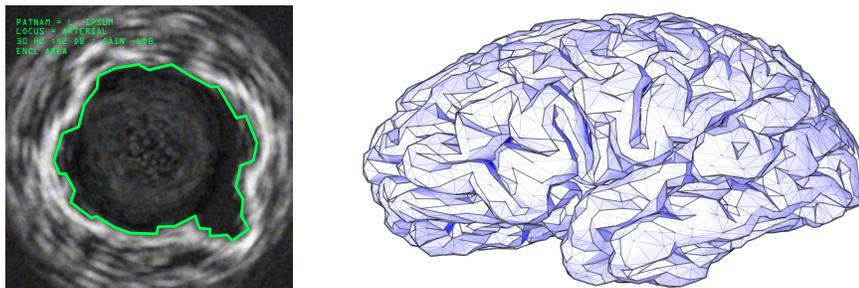


**Figure 1.** [LEFT] The boundary of the unobstructed cross section of an artery is approximated by a piecewise linear loop. [RIGHT] Medical imaging leads to volume-estimation problems for complex, nonconvex regions.

Similarly, one can use Gauss's theorem for computing volume based on a triangulation of the boundary. Here, again, medical imaging data provides a clear motivation, with nonconvex surfaces arising naturally (Figure 1[right]). Assume that a three-dimensional region has boundary sampled by a collection of points in $(x, y, z)$ coordinates. Triangulate this surface and label the vertices of each triangle $T^i$ as $(x_j^i, y_j^i, z_j^i)$ for $j = 1, 2, 3$, cyclically ordered so as to induce a positive orientation on the surface (outward pointing normals). By parameterizing each $T^i$ as a flat surface and integrating the 2-form $\frac{1}{3}(x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy)$, one obtains via Gauss's theorem:

$$V \approx \frac{1}{6} \sum_{i=1}^{N} x_1^i y_2^i z_3^i + x_2^i y_3^i z_1^i + x_3^i y_1^i z_2^i - x_3^i y_2^i z_1^i - x_2^i y_1^i z_3^i - x_1^i y_3^i z_2^i \, . \tag{4}$$

More can be done: centroids and moments in three-dimensions are again computable via combinatorial formulae on the vertices, and one can recover Archimedes' principle on buoyant force applying Gauss's theorem to $z \, dx \wedge dy$.

In these applications to area and volume, there is no clear advantage to using differential forms over vector calculus. The following pair of examples shows how thinking in terms of differential forms guides one to deeper applications to data [6].

**3. WHO'S IN THE LEAD WITH GREEN'S.** The classical application of Green's theorem for area has a modern update to problems of data coming from time series. The key insight is that the area 2-form $dx \wedge dy$ is *oriented* area. In the applications to area computations above, one ignores the orientation, since the desired output is positive. However, the global orientation retains information. In the case of time series data, this encodes *cyclic order*. Consider a pair of time series $x(t), y(t)$ in Figure 2[left]. One look at the data suggests that $x$ is a *leading* indicator and $y$ is *lagging*. Assuming these time series were periodic in $t$, they would generate a closed curve $\gamma = \partial D$ in the $(x, y)$-plane. The integral of $dx \wedge dy$ over $D$ would be positive when $x$ leads $y$, negative when $y$ leads $x$, and zero when they are in phase (or anti-phase).

For purely periodic functions, there are many ways to discern this order: correlations and Fourier coefficients come to mind. Those methods are, however, very sensitive to time axis reparameterizations. In contrast, many real-life phenomena are *cyclic* (roughly repetitive) without being rigidly *periodic*. Cardiac rhythms, musculo-skeletal movements exercised during a gait, population dynamics in closed ecosystems, business cycles, and more are examples of cyclic yet aperiodic processes.

The key insight is that the loop integral in (2) is independent of (oriented) time-reparameterization: no matter how you write $\gamma = \partial D$ as an oriented parametric curve $(x(t), y(t))$ the oriented area will be the same, even if there is some backtracking along the path. For discretized data, this means that the estimate from equation (3) is robust with respect to nonuniformities in sampling of points on $\gamma$.
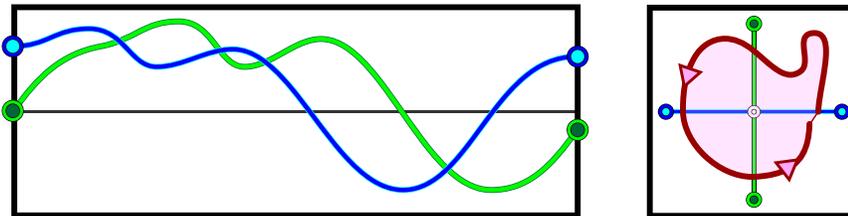


**Figure 2.** Left to right: two cyclic time series, $x(t)$ (starting positive) and $y(t)$ (starting at zero); $x$ approximately leads $y$; the corresponding parametric curve in the plane traces out a positive area, as measured by $x \, dy - y \, dx$, indicating leadership.

Here lies an excellent motivation for differential forms on $\mathbb{R}^n$: for multiple cyclic, noisy time series $\{x_i(t)\}_1^n$, the full leader-follower ordering can be derived from the integrals of $dx_i \wedge dx_j$ for all $i \neq j$. These integrals give a skew-symmetric matrix of nonparametric phase values. Further manipulations can reveal combinations of variables that are leading or trailing. The key to all of this is interpreting 2-forms on $\mathbb{R}^n$ as oriented projected area, and using Green's theorem as in (2) and (3).

**4. THE FORM OF SPHERICAL AREA.** Green and Gauss have put in an appearance: what, then, of Stokes? By analogy with the previous examples of area and volume, the three-dimensional Stokes's theorem would seem to be helpful in computing the area of a surface based on coordinate data of a piecewise-linear approximation to the boundary. This problem is natural in the context of estimating surface area on regions of the Earth based on boundary points (Figure 3). One sees specific examples in estimating the surface areas of countries or states, with border landmarks given in latitude and longitude, or the surface area of a connected component of the *Great Pacific Garbage Patch* or the polar ice caps.
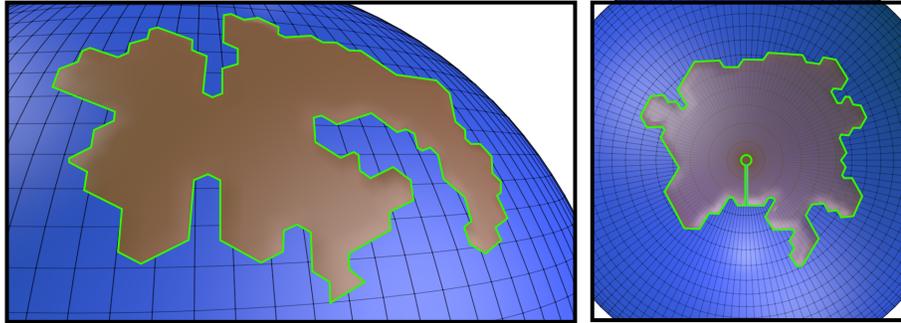
**Figure 3.** [LEFT] One can estimate the area of a region on the Earth's surface with piecewise-linear boundary. [RIGHT] One needs care with geographic and mathematical poles.

For the sake of exposition, assume that the Earth's surface is a sphere of radius $R$ and that spherical coordinates are used: $(\rho, \theta, \varphi)$ with $0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq \pi$. The area 2-form on the sphere of radius $R$ with standard (outward pointing) orientation is recognizable from the volume form in spherical coordinates: $R^2 \sin \varphi \, d\varphi \wedge d\theta$. In order for (1) to be applicable, one needs this form to be *exact*, the derivative of a 1-form. It is, since

$$R^2 \sin \varphi \, d\varphi \wedge d\theta = d(-R^2 \cos \varphi \, d\theta). \tag{5}$$

This tells us what to integrate over the boundary. Assume that a piecewise-linear approximation to the boundary of the spherical patch is given by $N$ segments $\gamma^i$ in the $(\theta, \varphi)$ plane with start points $(\theta_1^i, \varphi_1^i)$ and endpoints $(\theta_2^i, \varphi_2^i)$. These must collectively give a simple, closed, oriented curve. Integrating the 1-form of (5) over the straight-line segments $\gamma^i$, summing, and invoking Stokes yields a surface area $S$ with

$$S \approx -R^2 \sum_{i=1}^{N} (\theta_2^i - \theta_1^i) \frac{\sin \varphi_2^i - \sin \varphi_1^i}{\varphi_2^i - \varphi_1^i}. \tag{6}$$

As far as the authors can tell, this formula does not appear in the literature on geographic area estimation. Most references suggest performing an area-preserving projection to the Euclidean plane, then applying (3) to the projected boundary landmarks. One NASA JPL technical report [**2**] derives a number of exact and approximate formulae using techniques from spherical geometry. One such estimate (found on page 7) is very similar to (6) above, though replacing the quotient of $\varphi^i$ terms with an average value over the segment. The more traditional vector-field approach to Stokes's theorem does not seem to facilitate this application so readily as the forms-approach. Observing that $R^2 \sin \varphi \, d\varphi \wedge d\theta$ is exact is more natural than, say, trying to find a vector field in spherical coordinates that has curl equal to $\partial/\partial\rho$, so that the flux of the curl yields surface area.

**5. THE STARRY POLE.** All that remains is to do some examples and show students just how well this method of integrating a 1-form over the boundary works in practice. Perhaps the most compelling example is that of a polar ice cap. Let us suppose for simplicity that one has a round cap centered at the North Pole at an inclination angle of $0 < \varphi_* < \pi/2$, with boundary circle $\gamma$ discretized arbitrarily into segments.

Application of (6) presents two problems. The first difficulty is that the $\varphi^i$ terms are singular. This is not a surprise, as the terms in (6) were derived from an integral

assuming nonzero variation in $\varphi$. The integral is simpler when $\varphi = \varphi_*$ is constant. The second difficulty is that the answer is clearly wrong, since $\oint_\gamma -R^2 \cos \varphi_* \, d\theta = -2\pi R^2 \cos \varphi_* < 0$. The hypotheses of Stokes's theorem have not been respected. The 1-form in (5) is a multiple of $d\theta$ and is not well-defined at the poles, invalidating its use in Stokes's theorem. There are no difficulties in applying (6) to regions that do not encircle one of the poles. Indeed, this resolves the ambiguity in which of two surfaces on the sphere the loop $\gamma$ encloses.

What if one wants to apply this result to measuring the area of a region encircling a pole? For an oriented, simple, closed curve $\gamma$ that surrounds a pole of the sphere, a small improvement using ideas from contour integrals (see Figure 3[right]) shows that

$$S = R^2 \oint_\gamma (1 - \cos \varphi) d\theta \,. \tag{7}$$

This correction term permits a combinatorial formula for $\gamma$ discretized.

Here is our final argument for using forms. The existence of two poles of the sphere, each of which contributes a term to the integrand that is describable as an *index* of $+1$, permits a clean segue for the curious from calculus to the Euler characteristic, de Rham cohomology, and other exalted spheres [**1, 7**].

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