On the cohomology ring of no k-equal manifolds

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Abstract. The ring structures on the cohomologies of no k-equal subspace arrangements are computed. These rings turn out to be generated by the homologies in dimension k − 2 factored by the quadratic relations among them.

1. Introduction

In this note we deal with arrangements of linear subspaces in $\mathbb{R}^n$ called k-equal arrangements.

Definition 1. k equal arrangement consists of $\binom{n}{k}$ subspaces $\{x_{i_1} = x_{i_2} = \ldots = x_{i_k}\}$ and is denoted as $A_{n,k}$.

This definition makes sense over any field, though here only the real case will be treated. The arrangement $A_{n,2}$ is the arrangement of Weyl mirrors for $A_n$ (whence $A$).

The k-equal arrangements first studied by Bjorner and coauthors [5, 4, 3], drew a lot of attention lately. Natural objects from many points of view, they play an important role in various applications.

The complement to the arrangement $A_{n,k}$ is called the no k-equal manifold $M_{n,k}$. The Betty numbers of this manifold were computed by several authors, see [3]. The aim of this note is to determine the ring structure on $H^*(M_{n,k})$. The motivation for this research was the desire to mimic Kontsevich’s construction of universal Vassiliev invariant in the setting of Arnold’s Strange theory of plane curves, see [1]. To this end, the cohomology ring of $M_{n,3}$ would be useful.

Another remarkable property of the manifold $M_{n,3}$ is that it is a $K(\pi, 1)$ space (see [9]). The cohomologies of $M_{n,k}$’s with higher k’s could play some role in the study of immersion spaces without k-fold points (compare [8]).
The standard tool to calculate the cohomologies of the complement to an arrangement is to apply Goresky-McPherson-Vassiliev formulae for the homologies of the one point compactification of the arrangement itself, which are related to the cohomologies in question via the Alexander duality. On this route some information is being lost, like the ring structure on the cohomology. In some situations restrictions on the relative codimensions of flats allow to circumvent this problem, see [6, 7]. However, these restrictions do not hold in case of $A_{n,k}$.

The primary aim of this note is to describe the cohomology rings $H^*(M_{n,k})$. For the $k$-equal arrangements over complex numbers, the ring structure in cohomologies was found in [11], where the general approach of De Concini and Procesi and the freeness of the cohomologies proved in [3] was used. The general results [6, 7] also allow to recover the ring structure of $H^*(M_{n,k}^C)$.

It turns out that these $H^*(M_{n,k}^R)$ with the multiplication structure provided with the cup-product are quadratic rings: they are generated by the components of smallest positive dimension modulo some quadratic relations among them. Our main result is the Theorem 1 establishing the ring isomorphism between $H^*(M_{n,k}^R)$ and an explicitly described quadratic ring.

In the process of proof explicit bases for the cohomologies is constructed. Another family of bases was found in [2], spanned by essentially same combinatorial objects. The relation between these two families is not yet completely clear.

Some notations. We will use the shorthand $\mathbf{n}$ for the set $\{1, \ldots, n\}$. The space $\mathbb{R}^n$ is understood as the space of functions on $\mathbf{n}$. Similarly, for any subset $I \subset \mathbf{n}$ (the capital letters $I, J$ etc are reserved for subsets of $\mathbf{n}$), $\mathbb{R}^I$ is the set of functions on $I$. The indicator functions of the subset $I$ are denoted as $e_I \in \mathbb{R}^n$. In a similar fashion the manifolds $M_{I,k}$ are defined.

Fix the numbers $k, n$ (such that $3 \leq k \leq n$) for the rest of the paper.

2. String Posets

It will be convenient to work with certain partial orders $P$ on $\mathbf{n}$ (the elements of $\mathbf{n}$ will be referred to as indices). The relation given by $P$ will be denoted as $\succ$; $i \approx j$ means that both $i \succ j$ and $j \succ i$.

Only some specific posets will be of interest for us.

**Definition 2.** A poset $P$ is called string if there exists a height function $h$ such that

- $i \succ j$ implies $h(i) \geq h(j)$,
- $h(i) > h(j)$ implies $i \succ j$, and
- restriction of $P$ to any level set of $h$ is either empty or full.
To write down string posets we will use the following notation: the level sets of the height function \( h \) (called blocks) will be taken into \( [\ ] \) - or \( (\ ) \) -brackets, with the block \( [I] \), \( I = \{i_1, \ldots, i_s\} \) staying for the tuple of equivalent elements \( i_1 \approx i_2 \approx \ldots \approx i_s \), and the block \( (I) \) stays for a tuple of unrelated elements. A block consisting of a single element is, by convention, a \( (\ ) \)-block. The blocks will be written in the \( \succ \)-decreasing order, i.e. the elements in a block on the left are \( \succ \)-greater than the elements of a block on the right.

**Example 1.**
\[
(35)(2)[146] = \{3 \succ 2, 5 \succ 2 \succ 1 \approx 6 \approx 4\}.
\]

**Definition 3.** A string poset with no two \( (\ ) \)-blocks neighboring is called separated.

The posets \( (I) \ (J) \ (K) \), where \( I \cap J \cap K = n \) is a partition of \( n \) and \( |I| = k - 1 \) are called elementary.

A string poset can be extended by ordering the indices within each \( (\ ) \)-block according to the standard order on \( n \). The number of indices which are less than \( j \) with respect to thus extended ordering will be called the position of \( j \) and denoted as \( p(j) \).

All indices within a \([\ ]\)-block have the same position; within a \((\ )\)-block — different.

For any partial orders \( P_1, \ldots, P_k \) we will denote the transitive closure of \( P_1 \cup \ldots \cup P_k \) as \( P_1 \circ \ldots \circ P_k \).

Clearly, \( \circ \) is an associative and commutative product on the set of posets on \( n \).

### 3. Ring \( A(n,k) \)

Let \( F \) be the free exterior ring generated by the elementary posets, which we assume to have degree \( k - 2 \). For an (ordered) partition \( P = I \cap J \cap K \) of \( n \) with \( |I| = k - 2 \) we define an element \( i(P) \) of \( F^{k-2} \)
\[
i(P) = \sum_{t \in I} (-1)^{|I| - 1} (I - \{t\}) \ (J + \{t\}) \ (K) + \sum_{\kappa \in K} (-1)^{|J|} (I) \ [J + \{\kappa\}] \ (K - \{\kappa\}) \).
\]

The signs will become clear later.

Let \( I_1 \subset F \) be the ideal generated by the elements \( i(P) \), where \( P \) runs through all possible partitions \( P = I \cap J \cap K \) of \( n \) with \( |I| = k - 2 \). We set \( G = F/I_1 = \bigwedge (F^{k-2}/I_1^{k-2}) \). Further, we denote as \( I_2 \) the ideal in \( G \) generated by the (classes of) products \( P_1 \wedge P_2, P_1, P_2 \) elementary, such that the transitive closure of \( P_1 \cup P_2 \) has a \([\ ]\)-block of size at least \( k \).

The (skew-commutative) ring \( A = A(n,k) = G/I_2 \) is graded (as \( G \) is and the ideal \( I_2 \) is homogeneous) and has nontrivial components in dimensions which are multiples of \( k - 2 \) only. The algebra \( A(n,k) \) is in fact isomorphic to the cohomology ring of \( M_{n,k} \).
4. Posets and cells

The isomorphism \( w : A(n, k) \rightarrow H^*(M_{n,k}) \) between \( A \) and \( H^*(M_{n,k}) \) is defined as follows.

To each poset \( P \) on \( \{1, \ldots, n\} \) (not necessarily a string one) we associate a cell \( C(P) \) in \( \mathbb{R}^n = \mathbb{R}^n \):

**Definition 4.** The function \( x : \{1, \ldots, n\} \rightarrow \mathbb{R} \) is in \( C(P) \) iff the ordering on \( \{1, \ldots, n\} \) induced from \( \geq \) contains \( \succ \) (in other words, \( C(P) \) is given by the system of linear equations and inequalities obtained by substituting \( \succ \)'s and \( \approx \)'s by \( > \)'s and \( = \)'s correspondingly). The codimension of the cell \( C(P) \) is the number of \( [ \cdot ] \)-blocks less the number of \( \cdot \)-blocks and will be referred to as the rank of \( P \). The cells corresponding to elementary posets are called elementary as well.

**Lemma 1.** The boundary of an elementary cell belongs to \( A_{n,k} \).

**Proof.** Indeed,

\[
\partial C((I) \; [J] \; (K)) = \bigcup_{i \in I} C((I - \{i\}) \; [\{i\} + J] \; (K)) \bigcup \bigcup_{k' \in K} C((I) \; [J + \{k\}] \; (K - \{k'\}))
\]

\[\square\]

The tangent space to an elementary cell \( C((I) \; [J] \; (K)) \) is spanned by the vectors \( \partial_j, j \in I + K \) and \( \partial_I = \sum_{j \in I} \partial_j \), where \( \partial_\cdot = \partial/\partial x \). We orient the cell \( C(P) \) by declaring the frame ordered by the positions of the indices positive and co-orient it using the standard orientation of \( \mathbb{R}^n = \mathbb{R}^n \). The intersection pairing defines (by previous Lemma) a mapping

\[ w : F^{k-2} \rightarrow H^{k-2}(M_{n,k}). \]

**Proposition 1.** The mapping \( w \) extends to a ring homomorphism \( w : A \rightarrow H^*(M_{n,k}) \).

**Proof.** First, one has to check that the subspace of \( F^1 \) generated by \( i(P) \)’s is in the kernel of \( w \). This is immediate as \( i(P) \) just represents the oriented boundary of the \( (n - k + 2\)-dimensional) cell \( C((I) \; [J] \; (K)) \), where \( P = I \cup J \cup K \).

We define \( w \) on the product

\[ P_1 \land \ldots \land P_p \in F^{p(k-2)} \]

of \( p \) elementary posets as the class dual to the intersection of the corresponding elementary cells, or, equivalently, to the cell \( C(P_1 \circ \ldots \circ P_p) \), properly cooriented.
If $P = P_1 \circ \ldots \circ P_p$ has all $[\ ]$-blocks of size $k - 1$, then $P = (J_0) [I_1] (J_1) \ldots [I_p] (J_p)$ (some of $J$'s can be empty) and its decomposition as the product of elementary posets is unique up to a permutation of factors. Assume now that the factors $P_q, q = 1 \ldots p$ correspond to $[\ ]$-blocks (i.e. $P_q = (A_q) [I_q] (B_q)$). We coorient then the cell $C(P)$ by the $p(k - 2)$-frame $F_1, \ldots, F_p$, where $F_q$ is the frame coorienting $C(P_q)$.

Geometrically the cell $C(P)$ is just the transversal intersection of the cells $C(P_q), q = 1 \ldots p$ whence its boundary is in $A_{n,k}$. Together with the coorientation this defines a class in $H^p(M_{n,k})$ which we denote as $w(P) = w(P_1 \land \ldots \land P_p)$. Clearly, $w(P) = \cup_q w(P_q)$.

If the $\circ$-product of two elementary posets $P_1$ and $P_2$ has a $[\ ]$-block of size $k$ or more, then the corresponding cells do not intersect in $M_{n,k}$ and the product of the corresponding classes $w(P_1) \sim w(P_2)$ vanishes. Also, it is immediate that if $P_1 \circ \ldots \circ P_k$ has a $[\ ]$-block of size $k$ or more, than that holds for some $P_i, P_j, 1 \leq i, j \leq k$ as well.

Main result.

**Theorem 1.** The homomorphism $w : A \to H^+(M_{n,k})$ is an isomorphism.
5. Tidy posets

To prove Theorem 1 we pick a certain basis in \( A \). Its elements can be identified with the (classes of) tidy posets.

**Definition 5.** A separated poset \( P = (J_0) \ (I_1) \ (J_1) \ ... \ (I_s) \ (J_s) \) is called tidy if

- all \( [\ ] \)-blocks are of size \( k - 1 \), and
- for any \( k = 1, \ldots, s, \max\{i \in I_k \cup J_k\} \in J_k \). In words, the maximal index in any 2 consecutive blocks \( \ldots \ (I_k) \ (J_k) \ldots \) lies in the \( (\ ) \)-block \( (J_k) \).

The second condition implies, in particular, that all \( (\ ) \)-blocks, with exception, possibly, of the first one, are nonempty. If \( P \) is a tidy poset of rank \( s \), that is with \( s \ [\ ] \)-blocks (of size \( k - 1 \)), then \( P \) is clearly the \( \cup \)-product of \( s \) elementary posets \( P_k, k = 1, \ldots, s \) and \( C(P) \) is the transversal intersection of the elementary cells \( C(P_k) \).

**Theorem 2.** The classes of tidy posets form a basis in \( A \).

We start the proof with the following

**Lemma 2.** The classes of tidy posets span \( A \).

**Proof.** Obviously, \( A \) is generated by the classes of separated posets with all \( [\ ] \)-blocks of size \( k - 1 \). Let \( P = (J_0) \ (I_1) \ (J_1) \ ... \ (I_s) \ (J_s) \) be a separated poset with \( [\ ] \)-blocks of size \( k - 1 \). Then the class \( w(P) \) is the product of elementary classes \( w(P) = \prod_{k \leq s} w(P_k), P_k = (A_k) \ (I_k) \ (B_k) \), where \( A_k(B_k) \) is the union of all indices with positions less (greater) than the position of indices in \( [I_k] \). For the pair of blocks \( \ldots \ (I_k) \ (J_k) \ldots \) for which the condition of tidiness is violated, we say that its index is \( (k, m) \), where \( m \) is the size of \( A_k \) (the number of elements to the left from the \( [\ ] \)-block \( [I_k] \)). The index of a poset \( P \) is the lexicographic maximum of indices of its \( [\ ] \)-blocks violating the tidiness condition (or \( (0, 0) \) if the poset is tidy).

Let \( P = (J_0) \ (I_1) \ (J_1) \ ... \ (I_s) \ (J_s) \) be a non tidy poset of index \( (k, m) \), and \( P = P_1 \circ \ldots \circ P_s, P_p = (A_p) \ (I_p) \ (B_p), p = 1, \ldots, s \). Denote the maximal index in \( I_k \) as \( m_k \). Consider the element \( i(P) \in F_k^{-2} \), where \( P = A_k \amalg I_k \ominus (m_k) \amalg B_k \oplus (m_k) \). It is a sum of signed cells corresponding to one of the following posets:

- \((A_k) \ (I_k + \{i'\} - \{m\}) \ (B_k + \{m\} - \{i'\}) \), \( i' \in A_k \), or
- \((A_k - \{i'\}) \ (I_k + \{i'\} - \{m\}) \ (B_k + \{m\}) \), \( i' \in A_k \), or
This means that the class of $P_k$ in $A$ is a $\mathbb{Z}$-linear combination of classes of posets of the first two types. In $A$ the resulting expansion of $P$ consists of posets with either lesser $k$ (for products involving elementary posets of the first type) or equal $k$ and lesser $m$ (for products involving elementary posets of the second type). Therefore, the class of $P$ is a sum of classes of posets with lexicographically lesser indices. Inductively, $P$ is then in the $\mathbb{Z}$-span of tidy classes.

Let $T$ be the free $\mathbb{Z}$-module generated by the tidy classes, $T = \oplus_{p=t(k-2)} T_p$. One has the natural homomorphism (of $\mathbb{Z}$-modules) $T \to A$ and the composed mapping $c : T \to H^*(M_{n,k}), P \mapsto w(P)$.

Let $P = (J_0) \ [I_1] \ (J_1) \ldots [I_s] \ (J_s)$ be a tidy poset (of rank $s$). Denote by $m_k$ the maximal index in $J_k$ and consider the string poset $\tilde{P} = (J_0) \ [I_1 + \{m_1\}] \ (J_1 - \{m_1\}) \ldots [I_s + \{m_s\}] \ (J_s - \{m_s\})$. The corresponding cell $C(\tilde{P})$ is the transversal intersection of the subspaces of arrangement $A_{n,k}$. Near a point of $C(\tilde{P})$ the manifold $M_{n,k}$ is locally the product of $s$ punctured $\mathbb{R}^k$'s and of a linear space. We denote the homology class represented by the product of $s$ small spheres $S^{k-1}$ around the punctures in $\mathbb{R}^k$'s as $t(P)$ (the orientation is not essential here) and will be referring to such classes as local. It is clear, that this class is well defined (up to a sign). This extends to a homomorphism $t : T \to H^c(M_{n,k})$.

Now we define a $\mathbb{Z}$-bilinear form on $T$ as

$$\langle P, P' \rangle = (c(P), t(P')).$$

**Lemma 3.** The bilinear form $\langle \cdot, \cdot \rangle$ is nondegenerate.

**Proof.** Let $P = (J_0) \ [I_1] \ (J_1) \ldots [I_s] \ (J_s)$ be a tidy poset. The coefficient $\langle P, P' \rangle$ can be nonzero only for those tidy $P'$ of rank $s$, for which the cycle $T(P)$ intersects the cell $C(P')$. Clearly, $T(P)$ intersects transversally and at a single point the cell $C(P)$. As for further intersections, assume that the basis vectors $P$ of rank $s$ are numbered so that the corresponding vectors of sizes of $(\cdot)$-blocks, $(n_1, \ldots, n_s)$ are lexicographically ordered. Then one sees easily that $T(P)$ intersects (besides $C(P)$) only $C(P')$'s with lexicographically greater size vectors. Therefore, the matrix of $\langle \cdot, \cdot \rangle$ is triangular with $\pm 1$'s on the diagonal, which proves the Lemma.

**Corollary 1.** The mapping $c : T \to H^*(M_{n,k})$ is injective; moreover $w : T \to A$ is an isomorphism.

**Proof of Theorem 2.** Follows immediately from Lemma 2 and Corollary 1.
6. Proof of Main Theorem

To finish the proof of the Theorem 1, we have to establish the surjectivity of \( w : A \to H^*(M_{n,k}) \).

or, rather, the surjectivity of \( t : T \to H_*(M_{n,k}) \): we will show now that the local classes generate the homologies of \( M_{n,k} \).

**Proposition 2.** The local classes generate \( H_*(M_{n,k}) \).

**Proof.** We will apply an induction argument. For \( n \leq k \) the claim is trivial.

Consider the projection \( f : M_{n+1,k} \to M_{n,k} \) forgetting the \( (n+1) \text{st} \) coordinate. Unlike the braid case, this is not a locally trivial fibration. We denote by \( M_{n,k}^{++} \) the \( f \)-preimage of \( M_{n,k} \). The complement \( M_{n,k}^{++} - M_{n+1,k} \) is the union of elementary cells with the index \( n+1 \) in the [ ]-block intersected with \( M_{n,k}^{++} \). We denote by \( M(\tilde{P}) \) the submanifolds \( M_{n,k}^{++} \cap C(P) \), where \( P \) is an elementary poset on \( \{1, \ldots, n+1\} \) with the index \( n+1 \) in the [ ]-block. These submanifolds do not intersect.

The structure of \( M(\tilde{P}) \) is simple: the [ ]-block splits the configuration in \( M(\tilde{P}) \) into two no-k-equivalent subconfigurations. More precisely, if \( (1) [ ] \{ K \} \) is an elementary poset (with \( \# J = k; J \ni n+1 \)), then \( g_{I,K} : M(\tilde{P}) \to M_{I,k} \times R_+ \times R \times R_+ \times M_{K,k} \) given by

\[
g_{I,K} : (x_1, \ldots, x_n, 1) \mapsto (x_1, x_n, 1 - \max_{i \in I} x_i, x_{n+1}, \min_{k \in K} x_k, x_{n+1}, x_k)
\]

is a homeomorphism (if both \( I \) and \( K \) and not empty; otherwise one omits the factors pertaining to the empty subsets). The projection \( f \) takes the manifold \( M(\tilde{P}) \) into the cell \( C( (1) [ ] \{ J - \{n+1\} \} \{ K \} ) \subset A_{n,k-1} \).

Let \( \gamma \) be a piece-wise linear cycle representing an element of \( H_*(M_{n+1,k}) \).

Perturbing it if necessary, we can assume that all simplices of \( \gamma \) are transversal to the fibers of \( f \), and the projections of simplices of \( \gamma \) are transversal to cells \( C( (1) [ ] \{ J - \{n+1\} \} \{ K \} ) \subset A_{n,k-1} \).

As the cycle \( \gamma \) is compact, the numbers \( B_- = \inf \gamma \min x_i, B_+ = \sup \gamma \max x_i \) is finite. Consider the homotopy of \( \gamma \) in \( M_{n,k}^{++} \) shifting \( \gamma \) along the fibers of \( f \) down by \( B_+ - B_- \) (i.e. \( \gamma_t = \gamma - (B_+ - B_-) t e_{n+1}, 0 \leq t \leq 1 \)). The trace of this homotopy is a chain \( \Gamma \) in \( M_{n,k}^{++} \) but not in \( M_{n+1,k} \): it can intersect some of the manifolds \( M(\tilde{P}) \).

To remedy this, consider small tubular neighborhoods \( N(\tilde{P}) \) (of varying radius) of the manifolds \( M(\tilde{P}) \). The radii of these neighborhoods can be chosen sufficiently small and slowly varying to ensure that

- different neighborhoods do not intersect, and
- all simplices (of a simplicial subdivision of) \( \Gamma \) are transversal to submanifolds \( \partial N(\tilde{P}) \).
The second condition can be achieved as the transversality properties assumed for $\gamma$ imply that the simplices of $\Gamma$ are transversal to submanifolds $M(\tilde{P})$. The intersection $\Gamma_p = \Gamma \cap \partial N(\tilde{P})$ is a piecewise-smooth cycle. The chain $\Gamma - \cup P N(\tilde{P})$ expresses the class $c$ as a sum of class of $\gamma_1$ and of classes $\Gamma_{\tilde{P}}$.

The intersection of $\Gamma$ with $M(\tilde{P})$ is a piece-wise smooth cycle generating an element $h_{\tilde{P}} \in H_{s-k+2}(M(\tilde{P}))$. The boundary $\partial N(\tilde{P})$ of the tubular neighborhood of $M(\tilde{P})$ is diffeomorphic to $S^{k-2} \times M(\tilde{P})$. defining an $s$-dimensional element in $H_s(\partial N(\tilde{P})) \cong H_s(S^{k-2} \times H_s(M(\tilde{P})))$ which is evidently equal to $[S] \otimes h_{\tilde{P}}$ (here $[S]$ is the class of the fiber of the trivial fibration $\partial N(\tilde{P}) \to M(\tilde{P})$). By induction assumption, each element of $H_s(M(\tilde{P}))$ is generated by the local classes in $M(\tilde{P})$. The product of a local class in $H_*(M(\tilde{P}))$ with the class of the fiber is local either.

The cycle $\gamma_1$ is contained in the open subset of $M_{n+1,k}$ where $x_{n+1} < \min_{0 \leq i \leq n} x_i$. This subset is homeomorphic to $M_{n,k} \times \mathbb{R}$ and its homologies are, by induction assumption, generated by the local classes in $M_{n,k}$ and therefore in $M_{n+1,k}$.

Hence the local classes generate the homologies of $M_{n+1,k}$ and the result follows.

**END OF THE PROOF OF THEOREM 1:** By Proposition 2, $t$ is an isomorphism, whence, by the universal coefficients theorem, it follows that $w$ is an isomorphism as well.

### 7. Concluding Remarks

An immediate corollary of the fact that tidy classes form a basis of cohomologies of $M_{n,k}$ is the formula for the Betti numbers.

**Proposition 3.** The Betti numbers for $M_{n,k}$ are

$$\dim H_{l(k-2)}(M_{n,k}) = \sum_{i_1, \ldots, i_l} \binom{n}{i_1, \ldots, i_l} \binom{i_1 - 1}{k-1} \cdots \binom{i_l - 1}{k-1}$$

in dimensions which are multiples of $k - 2$ and $0$ elsewhere.

**Proof.** A tidy class $(J_0 | I_1 | J_1) \cdots [I_l | J_l]$ classes is uniquely defined by a) a choice of $l$ subsets $I_k \cup J_k, k = 1 \ldots l$ and b) choices of subsets $I_k$ within these sets with maximal elements deleted.

### References


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